Relatively Open Gromov-Witten Invariants for Symplectic Manifolds of Lower Dimensions

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Abstract. Let (X, ω) be a compact symplectic manifold, L be a Lagrangian submanifold and V be a codimension 2 symplectic submanifold of X, we consider the pseudoholomorphic maps from a Riemann surface with boundary $(\Sigma, \partial\Sigma)$ to the pair (X, L) satisfying Lagrangian boundary conditions and intersecting V. We study the stable moduli space of such open pseudoholomorphic maps involving the intersection data. Assume that $L \cap V = \emptyset$ and there exists an anti-symplectic involution ϕ on X such that L is the fixed point set of ϕ and V is ϕ-anti-invariant. Then we define the so-called "relatively open" invariants for the tuple (X, ω, V, ϕ) if L is orientable and $\dim X \leq 6$. If L is nonorientable, we define such invariants under the condition that $\dim X \leq 4$ and some additional restrictions on the number of marked points on each boundary component of the domain.

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Contents

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1 Introduction

Classical Gromov-Witten invariants $GW_{g,n}(X, A)$ for a symplectic manifold (X, ω) count with sign the (J, ν)-stable maps u from the compact Riemann surface $\Sigma_{g,n}$ of genus g with n marked points to the symplectic manifold (X, ω), representing the homology class $A \in H_2(X)$ and satisfying some constraints, where J is a ω-compatible or ω-tamed almost complex structure on X and ν is an inhomogeneous perturbation satisfying the J-holomorphic map equation $\bar{\partial}_J u = \nu$. Such invariants originate from Ruan's introducing Donaldson type invariants in symplectic category (see [R]). The rigorous mathematical foundation of GW invariants was first established by Ruan-Tian [RT1] for semi-positive symplectic manifolds in 1993. Then many mathematicians contribute to the development and completion of such theme (see [FO][KM][LiT][RT2][Si]). In many cases these numbers coincide with the enumerative invariants in algebraic geometry. In order to find effective ways of computing these invariants, Tian [Ti] first showed a rough description of studying the degeneration of rational curves under a symplectic degeneration, an operation of 'splitting the target'. Then Ionel-Parker [IP1][IP2] and Li-Ruan [LR] independently defined the so-called relative Gromov-Witten invariants with respect to a codimension 2 symplectic submanifold V and also established the symplectic connect sum formula for GW invariants.

For recent years, theoretic physicists have predicted the existence of enumerative invariants about pseudoholomorphic curves with Lagrangian boundary conditions by studying dualities for open strings (see [AKV][OV]). Roughly speaking, let $L \subset X$ be a Lagrangian submanifold, such invariants would count pseudoholomorphic maps from a Riemann surface with boundary $(\Sigma, \partial\Sigma)$ to (X, L), representing a relative class $d \in H_2(X, L)$, such that the boundary is mapped in the Lagrangian submanifold.

There are two main difficulties in defining such invariants: orientability of moduli space and codimension 1 bubbling-off phenomenon. We denote by $\mathcal{M}_{k,l}(\Sigma, L, d)$ the moduli space of such maps u with k distinct marked points in $\partial\Sigma$ and l distinct marked points in Σ, and its compactfication by $\overline{\mathcal{M}}_{k,l}(\Sigma, L, d)$. Therefore, we have canonical evaluation maps at marked points

$$evb_i : \overline{\mathcal{M}}_{k,l}(\Sigma, L, d) \to L, \quad i = 1, \cdots, k,$$
$$evi_j : \overline{\mathcal{M}}_{k,l}(\Sigma, L, d) \to X, \quad j = 1, \cdots, l.$$

Note, by [FOOO][Sil], that $\mathcal{M}_{k,l}(\Sigma, L, d)$ might be non-orientable. If L is relatively spin, then Fukaya *et al.* [FOOO] proved that the moduli space is orientable. Katz-Liu [KL][Liu]

defined an open invariant under the much restricted assumption that there exists an additional S^1-action on the pair (X, L). To define an orientation on the moduli space, they also assumed the Lagrangian submanifold L is orientable and (relatively) spin. The orientability of moduli space seems important for Katz-Liu's definition of enumerative invariants. However, J. Solomon [So] showed that even if L is not orientable, under the weaker assumption that L is 'relatively Pin$^{\pm}$' and some constraints on the number of boundary marked points, we still can obtain a canonical isomorphism from the orientation bundle of $\mathcal{M}_{k,l}(\Sigma, L, d)$ to the pullbacked line bundle from the orientation bundle of L^k by the evaluation map $\prod_i evb_i$. Although the moduli space $\mathcal{M}_{k,l}(\Sigma, L, d)$ may be nonorientable, the integral of $\det(TL)$-valued forms over $\mathcal{M}_{k,l}(\Sigma, L, d)$ would make sense.

On the other hand, we at present have no effective method to completely deal with the codimension 1 boundary of moduli space coming from the bubbling off discs. According to the discussion in [AKV], in order to define an invariant independent of other choices, it seems necessary to introduce some additional parameter on (X, L). For instance, the assumption in [Liu] that there exists an S^1-action is used to cancel the codimension 1 boundary of moduli space. In the paper [So], under the assumption that there exists an anti-symplectic involution

$$\phi : X \to X, \qquad \phi^*\omega = -\omega,$$

such that $L = \mathrm{Fix}(\phi)$, and among other technical conditions, Solomon constructed the open invariants for $\dim X \leq 6$ if L is orientable and, for $\dim X \leq 4$ even if L might be non-orientable. Note that, from the viewpoint of real algebraic geometry, the assumption that symplectic manifold admits an anti-symplectic involution is much natural. Actually, in that paper, Solomon also showed that, for the genus zero domain curve and strongly semi-positive real symplectic manifold of dimension no more than 6, his open invariants exactly coincide with twice of the Welschinger's invariants which can be regarded as a lower bound of the number of real rational curves in a real symplectic manifold(cf. [W1][W2][W3]). Independently, Cho [C], with different way of counting, also defined similar enumerative invariants for strongly semi-positive symplectic manifold with $\dim X \leq 6$, genus $g = 0$, and L being relatively spin.

The aim of this article is to show a definition of "relatively open Gromov-Witten invariants" which count stable maps $u : (\Sigma, \partial\Sigma) \to (X, L)$ from a compact Riemann surface with boundary whose images intersect with a codimension 2 symplectic submanifold $V \subset X$, satisfying Lagrangian boundary conditions. These invariants are designed for a preparation of establishing open symplectic sum formulas, which will be used to compute open Gromov-Witten invariants of a symplectic connect sum of two symplectic manifolds (X, L_X, V) and $(Y, L_Y, \overline{V})$. More concretely, the symplectic sum is an operation that first removes V and $\overline{V}$ from X and Y, respectively, and then combines them into a new symplectic manifold $X\#Y$ with symplectic structures matching on the overlap region. A stable map into the sum is expected to be a pair of stable maps into the two sides which match in the middle. So the first step is to count stable maps in one side X which record the intersection points with V and multiplicities. At present, we only construct such invariants under the additional assumptions that L is the fixed point set of an anti-symplectic involution, $\dim X \leq 6$ and $L \cap V = \emptyset$. However, for possible applications in the future, we will define the related moduli space and study the problem of orientability for more general situation. An axiomized formulation of the symplectic sum of open GW

invariants has appeared in [LiuY].

Similar to the absolute case in [IP1], before considering such stable maps we have to extend J and ν to the connect sum. The V-*compatibility* conditions imposed on the pair (J, ν) defined in Section 4 ensure that such an extension exists. However, these conditions do not always hold for generic (J, ν). Our relatively open invariants count stable maps for these special V-compatible pairs, which are different from the way of counting of the absolute open GW invariants. Given such a special V-compatible pair (J, ν), V is a J-holomorphic submanifold, and a J-holomorphic map $u : (\Sigma, \partial\Sigma) \to (V, L_V)$ (if $L_V = L \cap V \neq \emptyset$) into (V, L_V) is automatically a J-holomorphic map into (X, L). Therefore, some domain components (closed or open) of stable maps maybe mapped entirely into V. Such maps are not transverse to V and the moduli space of such maps may be of dimension larger than the expected dimension of $\mathcal{M}_{k,l}(\Sigma, L, d)$.

To avoid such non-transversal intersections, we restrict consideration to the so-called V-regular open stable maps which have no components mapped entirely into V. We note that such maps may intersect V at finite many points (interior or boundary intersection points) with multiplicity. According to the ordering of these intersection points, the space of V-regular stable maps separates into components labeled by pairs of vectors $(\mathbb{r}, \mathbb{s})$, where $\mathbb{r} = (r_1, \cdots, r_{\mathbb{k}})$ and $\mathbb{s} = (s_1, \cdots, s_{\mathbb{l}})$. The subscript $\mathbb{k}$ (resp. $\mathbb{l}$) denotes the number of boundary (resp. interior) intersection points and r_i (resp. s_j) denotes the multiplicity of the i^{th} (resp. j^{th}) boundary (resp. interior) intersection point. Then in Section 5, we will study the moduli space $\mathcal{M}^V_{k,l}(\Sigma, L, d)$ of such V-regular stable maps. We will prove that each irreducible component $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{k,l,\mathbb{k},\mathbb{l}}(X, L, d)$ of V-regular maps is a manifold with boundary whose dimension is expressed in (5.3).

In Sections 6 and 7, we consider the compactification of the moduli space of V-regular open maps—the space of V-stable open maps, of which each component is a subset of the closure of the V-regular space $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{k,l,\mathbb{k},\mathbb{l}}(\Sigma, L, d)$ in the stable moduli space $\overline{\mathcal{M}}_{k+\mathbb{k},l+\mathbb{l}}(\Sigma, L, d)$. We will show that the irreducible part of this compactification is a manifold with boundary. The key observation is that each sequence $\{u_n\}$ of V-regular open maps limits to a stable map u with additional restriction. For instance, if the image of a component of such a limit map lies entirely in V (or (V, L_V)), then along this component, we can find a section ξ of the normal bundle of V (or (V, L_V)), such that the elliptic equation $D^N\xi = 0$ holds, where D^N is the restriction of the linearization operator of $\bar{\partial}_{J,\nu}$ to the normal bundle of V (or (V, L_V)). Stable maps with this additional restriction are called V-stable maps. Each component of the space of V-stable maps is denoted by $\overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{k,l,\mathbb{k},\mathbb{l}}(\Sigma, L, d)$, which is the compactification of the space of V-regular open maps with frontier strata of codimension at least one. To analyze the convergence of sequences of V-regular maps, we use the renormalization technique similar to ones in [IP1] and [T].

Then we come to define relative invariants for the case $L \cap V = \emptyset$. Let us introduce some more delicate notations. Recall that the maps we consider are (j, J, ν)-holomorphic maps u from a bordered Riemann surface $(\Sigma, \partial\Sigma)$ with fixed conformal structure j to a symplectic-Lagrangian pair (X, L) satisfying Lagrangian boundary conditions and representing a fixed relative homology class $d \in H_2(X, L)$. We suppose the boundary of Σ has m components, *i.e.* $\partial\Sigma = \bigcup_{a=1}^m (\partial\Sigma)_a$ with each $(\partial\Sigma)_a \simeq S^1$. Moreover, we also require that the image of each boundary component represents a fixed homology class, *i.e.* $u|_{(\partial\Sigma)_a *}([(\partial\Sigma)_a]) = d_a \in H_1(L)$. Suppose that there are k_a marked points $z_{a1}, \cdots, z_{ak_a}$ on each boundary

component $(\partial\Sigma)_a$ and l marked points $w_1, \cdots, w_l$ and additional $\ell = \mathbb{l}$ intersecting marked points $q_1, \cdots, q_{\mathbb{l}}$ on Σ.[1] We reset

$$\mathbf{d} = (d, d_1, \cdots, d_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m},$$

$$\mathbf{k} = (k_1, \cdots, k_m), \quad |\mathbf{k}| = \sum_{a=1}^{m} k_a,$$

$$\mathbf{u} = (u, \vec{z}, \vec{w}, \vec{q}), \quad \vec{z} = (z_{ai}), \quad \vec{w} = (w_j), \quad \vec{q} = (q_\jmath).$$

Then we rewrite the moduli space of V-regular maps $\mathbf{u}$ by $\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d})$. The compactification $\overline{\mathcal{M}}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d})$ is the space of V-stable open maps.

There are some natural evaluation maps at those marked and intersection points

$$evb_{ai} : \overline{\mathcal{M}}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d}) \to L, \quad i = 1, \cdots, k_a, \ a = 1, \cdots, m,$$

$$evi_j : \overline{\mathcal{M}}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d}) \to X, \quad j = 1, \cdots, l,$$

$$evi^I_\jmath : \overline{\mathcal{M}}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d}) \to V, \quad \jmath = 1, \cdots, \mathbb{l}.$$

The first problem is the orientability of the moduli space. To get a more general result, we do not expect $\overline{\mathcal{M}}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d})$ to have a canonical orientation, which means that there is a canonical orientation on the orientation line bundle $\det(\overline{\mathcal{M}}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d}))$. Instead, we only want to construct an orientation on a modified line bundle

$$\det(T\overline{\mathcal{M}}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d})) \otimes \bigotimes_{a,i} (evb^*_{ai} \det(TL).$$

The conditions that ensure there exists such an orientation are that the Lagrangian submanifold L is "relatively $Pin^{\pm}$" (see Definition 8.1) and some restrictions imposed on the boundary marked points. In Section 8 we obtain the following result.

Theorem 1.1 *Assume L is relatively $Pin^{\pm}$ and fix a relative $Pin^{\pm}$ structure on (X, L). If L is not orientable, we assume $k_a \cong w_1(d_a) + 1 \mod 2$. If L is orientable, fix an orientation. Then the relatively $Pin^{\pm}$ structure on (X, L) and the orientations of L if it is orientable determines a canonical isomorphism*

$$\det(T\overline{\mathcal{M}}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d})) \xrightarrow{\sim} \bigotimes_{a,i} evb^*_{ai} \det(TL). \tag{1.1}$$

Remark. By the Wu relation [MS], if $n \le 3$ then L is always Pin^-.

Given the isomorphism (1.1), we can define the relatively open invariants as follows. Denote the images of marked points under evaluation maps by

$$x_{ai} = u(z_{ai}), \ y_j = u(w_j), \ \mathbb{q}_\jmath = u(q_\jmath) \ .$$

[1] To avoid some kind of bubbling, in this article, we require that if $\Sigma \simeq D^2$, $k > 0$.

Let $\Omega^*(L, \det(TL))$ denote differential forms on L with values in $\det(TL)$, and let $\Omega^*(X)$ (resp. $\Omega^*(V)$) denote ordinary differential forms on X (resp. V). Let $\alpha_{ai} \in \Omega^n(L, \det(TL))$, $a = 1, \cdots, m; i = 1, \cdots, k_a$, represent the Poincaré dual of the point x_{ai} in $H^n(L, \det(TL))$, which is the cohomology of L with coefficients in the flat line bundle $\det(TL)$. Let $\gamma_j \in \Omega^{2n}(X)$ represent the Poincaré dual of y_j for $j = 1, \cdots, l$. And let $\eta_\jmath \in \Omega^{2n-2}(V)$ represent the Poincaré dual of $\mathbb{q}_\jmath$ for $\jmath = 1, \cdots, \mathbb{l}$. Then we define

$$\begin{aligned} \mathscr{RN} &:= \mathscr{RN}(V, \mathbf{d}, \mathbf{k}, l, \mathbb{l}) \\ &= \int_{\overline{\mathcal{M}}^{V,\ \mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\mathbf{d})} \bigwedge_{a,i} evb^*_{ai}(\alpha_{ai}) \bigwedge_j evi^*_j(\gamma_j) \bigwedge_\jmath evi^{I*}_\jmath(\eta_\jmath)\ . \end{aligned} \tag{1.2}$$

The integral (1.2) makes sense because by Theorem 1.1, the integrand is a differential form taking values in the orientation line bundle of the V-stable moduli space. Denote by $\mu : H_2(X, L) \to \mathbb{Z}$ the Maslov index, and by g the genus of the closed Riemann surface $\Sigma \cup_{\partial\Sigma} \overline{\Sigma}$ which is the complex double of Σ. From the dimensional calculation (5.3) we know that the integral above vanishes unless

$$\begin{aligned} (|\mathbf{k}| + 2l)n + 2\ell(n-1) &= \mu(d) + n(1-g) + k \\ &\quad +2(l + \ell - \deg \mathbb{s}) - \dim Aut(\Sigma). \end{aligned} \tag{1.3}$$

where $\deg \mathbb{s} = \sum_{j=1}^{\mathbb{l}} s_j$.

In general, the integral (1.2) might depend on the choice of the forms α_{ai}, *etc.*, because the codimension 1 boundary strata would contribute to the integral. To prove the invariance of the integral (1.2), we suppose that there exists an anti-symplectic involution ϕ on X, *i.e.* $\phi^*\omega = -\omega$, such that $L = \mathrm{Fix}(\phi)$ and the restriction of ϕ to V is also an involution on the submanifold. Furthermore, suppose Σ is biholomorphic to its conjugation $\bar{\Sigma}$, *i.e.* there exists an anti-holomorphic involution $c : \Sigma \to \Sigma$. Denote by $\mathcal{J}_\omega$ the set of ω-tamed almost complex structures on X. Let $\mathcal{P}$ denote the set of J-anti-linear inhomogeneous perturbation terms. Define

$$\mathcal{J}_{\omega,\phi} := \{J \in \mathcal{J}_\omega |\ \phi^* J = -J\}.$$

Now for each $J \in \mathcal{J}_{\omega,\phi}$, we denote by $\mathcal{P}^J_{\phi,c}$ the set of $\nu \in \mathcal{P}$ satisfying $d\phi \circ \nu \circ dc = \nu$. Denote

$$\mathbb{J} := \{(J, \nu) | J \in \mathcal{J}_\omega, \nu \in \mathcal{P}\},$$

$$\mathbb{J}_\phi := \{(J, \nu) | J \in \mathcal{J}_{\omega,\phi}, \nu \in \mathcal{P}^J_{\phi,c}\} \subset \mathbb{J}.$$

Fix $(J, \nu) \in \mathbb{J}^V_\phi$. Thus, from the V-stable (J, ν)-holomorphic map $u : (\Sigma, \partial\Sigma) \mapsto (X, L)$ we can define its conjugate V-stable (J, ν)-holomorphic map $\tilde{u} = \phi \circ u \circ c$ representing the homology class $[\tilde{u}] = -\phi_* d$, simply denoted by $\tilde{d}$. Denote $\tilde{\mathbf{d}} = (\tilde{d}, d_1, \cdots, d_m)$. So we have an induced map

$$\phi' : \overline{\mathcal{M}}^{V,\ \mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d}) \to \overline{\mathcal{M}}^{V,\ \mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X, L, \tilde{\mathbf{d}})$$

given by

$$\mathbf{u} = (u, \vec{z}, \vec{w}, \vec{q}) \mapsto \tilde{\mathbf{u}} = (\tilde{u},\ (c|_{\partial\Sigma})^{|\mathbf{k}|}(\vec{z}),\ c^l(\vec{w}),\ c^\ell(\vec{q})).$$

Define

$$\Omega^*_\phi(X) := \{\gamma \in \Omega^*(X) |\ \phi^*\gamma = \gamma\},$$

$$\Omega^*_\phi(V) := \{\eta \in \Omega^*(V) |\ \phi^*\eta = \eta\}.$$

Now in the integral (1.2) we take the forms $\gamma_j \in \Omega^{2n}_\phi(X)$ and $\eta_\jmath \in \Omega^{2n-2}_\phi(V)$.

Then we study the lower dimensional cases, *i.e.* $n \leq 3$. If $d = \tilde{d}$, then using the method in this paper (see Sections 8–11), we can show that the integrals (1.2) are invariants of the tuple (X, ω, V, ϕ). Roughly speaking, we can show that the map ϕ' is an induced involution on the V-stable moduli space $\overline{\mathcal{M}}^{V,\,\$}_{\mathbf{k},l,\mathbb{1}}(X, L, \mathbf{d})$. And using the isomorphism (1.1), we can assign a sign to each V-stable map (see Section 9). Then we can prove that on each codimension 1 boundary stratum, the induced involution ϕ' is fixed point free and orientation reversing, *i.e.* changing the sign of each disc-bubbling stable map. Therefore, the contributions from the codimension 1 boundary of $\overline{\mathcal{M}}^{V,\,\$}_{\mathbf{k},l,\mathbb{1}}(X, L, \mathbf{d})$ to the integrals (1.2) are eliminated. That implies the invariance.

However, for general situation, the relative homology class d might not be ϕ-anti-invariant, then ϕ' might not be a map from the space of V-stable maps to itself. Thus the numbers $\mathscr{RN}$ are not always well-defined invariants. So we have to modify the definition above.

The idea is to take all related moduli space together to eliminate the contributions from the codimension 1 boundaries to the integral. In fact, such boundary-canceling method has been used in [C].

For a homology class $\alpha \in H_2(X)$ and a relative class $\beta \in H_2(X, L)$, we say $\beta_{\mathbb{C}} = \mathbb{d}$ if there is a holomorphic map u of class β such that its complex double $u_{\mathbb{C}}$ represents the homology class α. In this sense, we may say $\alpha = \beta_{\mathbb{C}}$ is the doubling of β. Note that $\beta_{\mathbb{C}} = (\tilde{\beta})_{\mathbb{C}}$. Denote by $\mathbb{d} = d_{\mathbb{C}}$ the homology class in $H_2(X)$ which is the doubling of d, and by $\bar{\alpha} = (\alpha_{ai})$, $\bar{\gamma} = (\gamma_j)$, $\bar{\eta} = (\eta_\jmath)$. Then we define

$$\begin{aligned}
\mathcal{I} := & \ \mathcal{I}^{V,\,\$}_{X,\phi,g,\mathbb{d},\mathbf{k},l}(\bar{\alpha}, \bar{\gamma}, \bar{\eta}) \\
= & \sum_{\forall \beta:\, \beta_{\mathbb{C}} = \mathbb{d}} \quad \sum_{\substack{\gamma_j:\, [\gamma_j] = \mathrm{PD}(\xi_j),\, \eta_\jmath:\, [\eta_\jmath] = \mathrm{PD}(\lambda_\jmath), \\ \forall (\vec{x}, \vec{\xi}, \vec{\lambda}) \in \mathcal{R}}} \\
& \int_{\overline{\mathcal{M}}^{V,\,\$}_{\mathbf{k},l,\mathbb{1}}(X,L,\bar{\beta})} \bigwedge_{a,i} evb^*_{ai}(\alpha_{ai}) \bigwedge_j evi^*_j(\gamma_j) \bigwedge_\jmath evi^{I*}_\jmath(\eta_\jmath)\ ,
\end{aligned} \tag{1.4}$$

where $\mathcal{R}$ is the space of real configurations (see Section 10, (10.6) for detailed definition), $\bar{\beta} = (\beta, d_1, \cdots, d_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m}$, $\mathrm{PD}(\cdot)$ is the Poincaré dual of the point class. Now, we state our main result.

Theorem 1.2 *Assume that* $L \cap V = \emptyset$*, and that* $\dim X \leq 6$*, if* L *is not orientable, we assume that* $\dim X \leq 4$ *and* $k_a \cong w_1(d_a) + 1 \mod 2$*. If each moduli space* $\overline{\mathcal{M}}^{V,\,\$}_{\mathbf{k},l,\mathbb{1}}(X, L, \bar{\beta})$ *generically contains no* ϕ*-multiply covered pseudoholomorphic disc*[2]*, then the integers* $\mathcal{I} = \mathcal{I}^{V,\,\$}_{X,\phi,g,\mathbb{d},\mathbf{k},l}$ *are independent of the generic choice of pair* $(J, \nu) \in \mathbb{J}^V_\phi$*, the choice of conformal structure* j*, or the choice of forms* $\alpha_{ai} \in \Omega^n(L, \det(TL))$*,* $\gamma_j \in \Omega^{2n}_\phi(X)$*,* $\eta_\jmath \in \Omega^{2n-2}_\phi(V)$*. Therefore,* $\mathcal{I}$ *are invariants of the tuple* (X, ω, V, ϕ)*.*

[2] A nonconstant map $u : (\Sigma, \partial\Sigma) \to (X, L)$ is called *ϕ-multiply covered* or *not ϕ-somewhere injective* if there exists a regular point $z \in \Sigma$ such that either $u(z) \in u(\Sigma \setminus \{z\})$ or $u(z) \in \mathrm{Im}(\phi \circ u)$.

Remark. In particular, if $d = \tilde{d}$, $\mathscr{RN} := \mathscr{RN}(V, \mathbf{d}, \mathbf{k}, l, \mathbb{1})$ are invariants of the tuple (X, ω, V, ϕ). $\mathscr{RN}$ can be regarded as an extension of Solomon's definition of open GW invariants $N_{\Sigma,\mathbf{d},\mathbf{k},l}$ to the relative case.

Remark. In the proof of Theorem 1.2 in Section 11, we use the assumption that there is no ϕ-multiply covered pseudoholomorphic disc. Actually, we believe, by using the virtual cycle techniques to the case of open maps or defining the equivariant Kuranish structure like what Solomon has done in section 7 of [So], that such assumption is not necessary and the Theorem 1.2 holds for general case.

In Section 8, we deal with the problem of orientation and derive the conclusion of Theorem 1.1. Then in Section 9, we assign a sign to each V-stable map and study how the induced action by involution changes the sign. The last two sections devote to the proof of Theorem 1.2. The content of Sections 2 and 3 is preparation for succeeding discussion, in Section 2, we show the definition of stable open (J, ν)-holomorphic maps and describe the moduli space of such maps. And we outline the constructions of classical GW, relative GW and Solomon's open GW invariants (with somewhat more general formulation) in Section 3. For the convenience of the reader, in Appendix we review some definitions and important conclusions in [So] about the orientation of determinant of real linear Cauchy-Riemann operator.

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2 Stable open pseudo-holomorphic maps

In this section, we show the definition of stable open (J, ν)-holomorphic maps, and describe the moduli space of such maps. The domains of such stable open maps are bordered Riemann surface *i.e.* compact Riemann surface with boundary, smooth or allowing nodal singularities. The boundary is mapped in a Lagrangian submanifold.

In the following, both marked points and double points (or singular points or nodes) are called *special points*. And we always fix a conformal structure on the open domain curve Σ, which is a bordered compact Riemann surface. We say such curve is of genus g if the closed Riemann surface $\Sigma \cup_{\partial\Sigma} \bar{\Sigma}$, which is the complex double of Σ and also denoted by $\Sigma_{\mathbb{C}}$ (see [AG]), is of genus g. Denote the genus of the closed surface $\Sigma/\partial\Sigma$ by g_0.

Assume that there are altogether m boundary components, *i.e.* $\partial\Sigma = \bigcup_{a=1}^{m} (\partial\Sigma)_a$. We say Σ is of topological type (g_0, m).

Definition 2.1 *An automorphism of a bordered Riemann surface Σ is a diffeomorphism $\varphi : \Sigma \to \Sigma$ preserving the conformal structure and the ordering of the boundary compo-*

nents. The set of all automorphism of Σ *is denoted by* $\mathrm{Aut}(\Sigma)$*. We say* Σ *is stable if* $\mathrm{Aut}(\Sigma)$ *is finite.*

Denote by $\Sigma_{\mathbb{C}}$ the complex double of Σ, which is a closed Riemann surface with an antiholomorphic involution σ. If $\varphi : \Sigma \to \Sigma$ is an automorphisim, then its complex double $\varphi_{\mathbb{C}} : \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ is an automorphism of $(\Sigma_{\mathbb{C}}, \sigma)$. This provides a natural inclusion $\mathrm{Aut}(\Sigma) \subset \mathrm{Aut}(\Sigma_{\mathbb{C}}, \sigma)$. The following statements are equivalent:

- Σ is stable, *i.e.*, $\mathrm{Aut}(\Sigma)$ is finite.
- $\Sigma_{\mathbb{C}}$ is stable.
- The genus $g = 2g_0 + m - 1$ of $\Sigma_{\mathbb{C}}$ is bigger than one.
- The Euler characteristic $\chi(\Sigma) = 2 - 2g_0 - m$ of Σ is negative.

Definition 2.2 *A prestable bordered Riemann surface with fixed conformal structure is either a smooth bordered Riemann surface or the union of a smooth bordered Riemann surface with finite sphere and disc bubbles.*

In this paper, we only consider the prestable bordered Riemann surfaces with fixed conformal structure, and simply call them prestable bordered Riemann surfaces.

Definition 2.3 *An automorphism of a prestable bordered Riemann surface* Σ *of topological type* (g_0, m) *with* $(\mathbf{k}, l)$ *marked points*

$$(\Sigma, \partial\Sigma, \vec{z}, \vec{w}) = (\Sigma, \{(\partial\Sigma)_a\}_{a=1}^m, \{z_{a1}, \cdots, z_{ak_a}\}_{a=1}^m, w_1, \cdots, w_l)$$

is an automorphism of Σ *preserving all marked points. The set of all automorphism is still denoted by* $\mathrm{Aut}(\Sigma)$*. A prestable* $(\mathbf{k}, l)$*-marked bordered Riemann surface* Σ *is stable if* $\mathrm{Aut}(\Sigma)$ *is finite.*

Definition 2.4 *A* $(\mathbf{k}, l)$*-marked bordered bubble domain curve* Σ *of type* (g_0, m)*, which is also called a prestable* $(\mathbf{k}, l)$*-marked bordered Riemann surface and where the vector* $\mathbf{k} = (k_1, \cdots, k_m)$*, is a finite connected union of smooth oriented compact surfaces* Σ_i*, at least one surface with boundary, joined at interior or boundary double points together with* k_a *distinct marked points in the boundary* $(\partial\Sigma)_a$*,* $a = 1, \cdots, m$*, and* l *distinct interior marked points, none of which are double points. The* Σ_i*, with their special points, are of two types:*
(1) *stable components, and*
(2) *unstable components, which are unstable sphere bubbles or unstable disc bubbles.*
And there must be at least one stable component.

There exists a natural stablization map

$$\mathrm{st} : \Sigma \to \widehat{\Sigma} \tag{2.1}$$

that collapses the unstable components to points, thus we get a connected domain $\widehat{\Sigma} = \mathrm{st}(\Sigma)$ which is a stable genus g open curve.

Bordered bubble domains can be constructed from a stable bordered Riemann surface Σ_0 by replacing points by finite chains of 2-spheres or 2-discs or their combination. Alternatively, they can be obtained from a smooth Riemann surface Σ_0 by pinching a set

of nonintersecting embedded circles in the interior of Σ_0 and (or) a set of half-circles in Σ_0 with centers in the boundary $\partial\Sigma_0$. The latter viewpoint can be formalized as follows. Assume that there are b_a double points on each boundary component $(\partial\Sigma)_a$, $a = 1, \cdots, m$, and there are d interior double points. Denote by $\mathbf{b} = (b_1, \cdots, b_m)$.

Definition 2.5 *A resolution of (g_0, m)-type $(\mathbf{k}, l)$-marked bordered bubble domain Σ with $(\mathbf{b}, d)$-double points is a smooth bordered Riemann surface with genus g, d embedded circles γ_c in the interior part and b_a embedded half-circles γ_{ah} with centers in each $(\partial\Sigma_0)_a$, $a = 1, \cdots, m$, (any two of distinct circles or half-circles are disjoint), and the $(\mathbf{k}, l)$-marked points are apart from γ_{ah} and γ_c, together with a resolution map*

$$\mathscr{R} : \Sigma_0 \to \Sigma$$

which respects orientation and marked points, takes each γ_c (resp. γ_{ah}) to an interior (resp. boundary) double point of Σ, and restricts to a diffeomorphism from the complement of $\bigcup \bar{\gamma}_{ah} \cup \bar{\gamma}_c$ in Σ to the complement of the double points, where $\bar{\gamma}_{ah}$ (resp. $\bar{\gamma}_c$) denotes the closure of half-disc (resp. disc) contained by γ_{ah} (resp. γ_c).

We next define (J, ν)-holomorphic maps from bubble domains. Such maps depend on the choice of an ω-compatible (tamed) almost complex structure J and an inhomogeneous perturbation ν to the Cauchy-Riemann equation. Let π_i, $i = 1, 2$, denote the projection from $\Sigma \times X$ to the i^{th} factor and let π' denote the restriction of π_i to $\partial\Sigma \times L$. We define the inhomogeneous term to be the section

$$\nu \in \Gamma(\Sigma \times X, \mathrm{Hom}(\pi_1^* T\Sigma, \pi_2^* TX))$$

such that

(1) ν is (j_Σ, J)-anti-linear: $\nu \circ j_\Sigma = -J \circ \nu$;

(2) $\nu|_{\partial\Sigma\times L}$ carries a sub-bundle ${\pi'}_1^* T\partial\Sigma \subset \pi_1^* T\Sigma$ to the sub-bundle ${\pi'}_2^*(JTL) \subset \pi_2^* TX$.

Denote by $\mathcal{P}$ the set of all such inhomogeneous terms. Let $\mathcal{J}$ denote the space of such pairs (J, ν).

Definition 2.6 *A (J, ν)-holomorphic open map from a bordered bubble domain curve $(\Sigma, \partial\Sigma)$ with complex structure j_Σ is a map*

$$u : (\Sigma, \partial\Sigma) \to (X, L)$$

such that, on each component Σ_i of Σ, u is a smooth solution of the inhomogeneous Cauchy-Riemann equation

$$\bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j_\Sigma) = \nu(\cdot, u(\cdot)), \tag{2.2}$$

or equivalently,

$$\bar{\partial}_{(J,\nu)} u = 0$$

where $\bar{\partial}_{(J,\nu)}$ denotes the perturbed nonlinear elliptic operator $\frac{1}{2}(d + J \circ d \circ j_\Sigma) - \nu$. In particular, $\bar{\partial}_J u = 0$ on each unstable bubble.

Remark. It is not difficult to show that the operator $\bar{\partial}_{(J,\nu)}$ gives rise to an elliptic boundary value problem. We refer to the Lemma 4.1 in [So].

The symplectic area of the image is the number

$$A(u) = \int_{u(\Sigma)} \omega = \int_{\Sigma} u^*\omega \tag{2.3}$$

which depends only on the homology class of the curve modulo its boundary. And the energy of u is

$$E(u) = \frac{1}{2}\int_{\Sigma} |du|^2_{J,\mu} d\mu \tag{2.4}$$

where $|\cdot|_{J,\mu}$ is the norm defined by the metric on X determined by J and the metric μ on the domain. For $(J,0)$-holomorphic maps, $E(u) = A(u)$.

We also define a *modified energy* $\mathbb{E}$ componentwise

$$\mathbb{E}(u_i) = \begin{cases} 1 + \frac{1}{2}\int_{\Sigma_i} |du_i|^2_{J,\mu} d\mu, & \Sigma_i \text{ is stable;} \\ \frac{1}{2}\int_{\Sigma_i} |du_i|^2_{J,\mu} d\mu, & \Sigma_i \text{ is unstable.} \end{cases} \tag{2.5}$$

And

$$\mathbb{E}(u) = \sum_i \mathbb{E}(u_i). \tag{2.6}$$

Now we come to the definition of *stable* map

Definition 2.7 *A (J,ν)-holomorphic map u is stable if each of its component maps $u_i = u|_{\Sigma_i}$ has positive modified energy i.e. $\mathbb{E}(u_i) > 0$ for each i.*

That is to say, either each component Σ_i of the domain is a stable curve, or else the images of $(\Sigma_i, \partial\Sigma_i)$ carries a nontrivial homology class. We have

Lemma 2.1 *Let (X,ω) be a compact symplectic manifold and L be a compact Lagrangian submanifold. Then*

(1) *every (J,ν)-holomorphic map has $\mathbb{E}(u) \geq 1$.*

(2) *There exists a constant $\hbar > 0$ such that for every component u_i of every stable (J,ν)-holomorphic map $u : (\Sigma, \partial\Sigma) \to (X, L)$ with Lagrangian boundary condition, we have*

$$\mathbb{E}(u_i) > \hbar \tag{2.7}$$

Proof. (1) The conclusion is direct since at least one component is stable.

(2) On the stable components, we have $\mathbb{E}(u_i) \geq 1$. On the unstable components, since u_i is a J-holomorphic map, the Proposition 4.1.4 of [McS] implies there exists $\hbar > 0$ depending only on (X, J) such that

$$\mathbb{E}(u_i) = E(u_i) = \frac{1}{2}\int_{\Sigma_i} |du_i|^2_{J,\mu} d\mu > \hbar,$$

provided the images of $(\Sigma_i, \partial\Sigma_i)$ carries a nontrivial homology class. Otherwise, if u_i represents a trivial homology class, then $\mathbb{E}(u_i) = E(u_i) = A(u_i) = \omega \cap [u_i(\Sigma_i)] = 0$, contrary the definition of stable map. $\square$

The following Gromov Convergence Theorem, which is also called Compactness Theorem, is the important fact about (J,ν)-holomorphic maps. It means that every sequence of (J,ν)-holomorphic maps from a smooth (bordered) domain has a subsequence which converges modulo automorphism to a stable map. Various forms of such convergence theorem have been proved for closed curves (c.f. [G], [PW], [RT1], [Y], *etc.*). For open curves, we refer to [Liu], [McS].

Theorem 2.1 (Bubble Convergence). *Let (X,ω) be a compact symplectic manifold, $L\subset X$ be a Lagrangian submanifold. Given any sequence $\{u_j\}$ of $(\mathbf{k},l)$-marked (J_j,ν_j)-holomorphic maps from a bordered Riemann surface Σ_0 satisfying Lagrangian boundary conditions, with $\mathbb{E}(u_j)<E_0$ and $(J_j,\nu_j)\to(J,\nu)$ in C^k, $k\geq 0$, then we can obtain a subsequence and*

(1) a $(\mathbf{k},l)$-marked bordered bubble domain Σ with resolution $\mathscr{R}:\Sigma_0\to\Sigma$, and

(2) automorphisms φ_j of Σ_0 preserving the orientation and the marked points, such that the modified subsequence $\{u_j\circ\varphi_j\}$ converges to a limit

$$\Sigma_0 \xrightarrow{\mathscr{R}} \Sigma \xrightarrow{u} X$$

where u is a stable (J,ν)-holomorphic map. This convergence is in C^0, and in C^k on compact sets not intersecting the collapsing curves γ_{ah} and γ_c of the resolution $\mathscr{R}$, and the area (2.3) and energy (2.6) are preserved in the limit.

Given a bordered bubble domain curve with fixed complex structure $(\Sigma,\partial\Sigma)$. Denote the space of equivalence classes of stable $(\mathbf{k},l)$-marked open pseudoholomorphic maps with Lagrangian boundary conditions representing the homology class $\mathbf{d}$ by

$$\overline{\mathcal{M}}_{\Sigma,\mathbf{k},l}(X,L,\mathbf{d}) \quad \text{or simply} \quad \overline{\mathcal{M}}_{\mathbf{k},l}(X,L,\mathbf{d}).$$

Definition 2.8 *We say a map $u:(\Sigma,\partial\Sigma)\to(X,L)$ is* multiply covered *if there does not exist a point $z\in\Sigma$ such that*

$$du(z)\neq 0, \quad \text{and} \quad u(z)\notin u(\Sigma\setminus\{z\}).$$

A map is called irreducible *or* somewhere injective *if it is not multiply covered.*

The multiply covered maps are often singular points in the moduli space $\mathcal{M}_{\mathbf{k},l}(X,L,\mathbf{d})$. For keeping the paper in suitable length, we will essentially avoid dealing with these maps in the present paper, and the author plans to treat in another paper.

Let $\overline{\mathcal{M}}_{\mathbf{k},l}(X,L,\mathbf{d})^*$ be the space of irreducible stable open maps. The following theorem is direct from Theorem 2.1 and the index theorem (see [FOOO][Liu][McS])

Theorem 2.2 *(1)$\overline{\mathcal{M}}_{\mathbf{k},l}(X,L,\mathbf{d})$ is compact and Hausdorff, and there exists evaluation map*

$$\mathrm{ev}:\overline{\mathcal{M}}_{\mathbf{k},l}(X,L,\mathbf{d})\longrightarrow X^l\times L^{|\mathbf{k}|} \tag{2.8}$$

(2)For generic $(J,\nu)\in\mathcal{J}$, $\overline{\mathcal{M}}_{\mathbf{k},l}(X,L,\mathbf{d})^*$ is a manifold with boundary (corners) of dimension

$$\dim\overline{\mathcal{M}}_{\mathbf{k},l}(X,L,\mathbf{d})^*=\mu(d)+n(1-g)+k+2l-\dim Aut(\Sigma). \tag{2.9}$$

(3)For generic $(J,\nu)\in\mathcal{J}$, if $\overline{\mathcal{M}}_{\mathbf{k},l}(X,L,\mathbf{d})$ is irreducible, then the image of $\overline{\mathcal{M}}_{\mathbf{k},l}(X,L,\mathbf{d})$ under ev *is a fundamental chain.*

3 Some invariants for symplectic manifolds

In this section, we show a rough description of the constructions of GW, relative GW and Open GW invariants. We will not repeat the detailed proofs since they can be found in references listed in the introduction part.

3.1 Gromov-Witten invariants

Let $\Sigma_{g,n}$ be a compact Riemann surface of genus g with n marked points, (X,ω) be a closed symplectic manifold, we denote by $\mathcal{J}_\omega$ the set of ω-tamed almost complex structures on X, and denote by $\mathcal{J}$ the set of all pairs (J,ν) with $J \in \mathcal{J}_\omega$. For a second homology class $A \in H_2(X)$, we denote by $\mathcal{M}_{g,n}(X,A)$ the moduli space of (J,ν)-holomorphic curves $u : \Sigma_{g,n} \to X$ representing A. Its Gromov compactification is denoted by $\overline{\mathcal{M}}_{g,n}(X,A)$. For generic (J,ν), $\overline{\mathcal{M}}_{g,n}(X,A)$ is a closed smooth manifold. If we denote the Deligne-Mumford compactification of the domain curves by $\overline{\mathcal{M}}_{g,n}$, then we have a canonical projection

$$\pi : \overline{\mathcal{M}}_{g,n}(X,A) \to \overline{\mathcal{M}}_{g,n},$$

which is called "stabilization" defined by collapsing all unstable components of the domain curve. Moreover, we have canonical evaluation maps

$$ev_i : \overline{\mathcal{M}}_{g,n}(X,A) \to X, \quad ev_i(u) = u(x_i), \quad i = 1, \cdots, n,$$

where $x_1, \cdots, x_n$ are n marked points on the domain. So we have a map

$$\pi \times \mathbf{ev} : \overline{\mathcal{M}}_{g,n}(X,A) \to \overline{\mathcal{M}}_{g,n} \times X^n. \tag{3.1}$$

If we assume that the moduli space has the expected dimension[3], *i.e.* by index theorem

$$\dim\overline{\mathcal{M}}_{g,n}(X,A) = 2c_1(A) + (\dim X - 6)(1-g) + 2n,$$

then the Gromov-Witten invariant is the homology class of the image of this map for generic (J,ν). Or, in other words, if we choose differential forms α_i and β such that $[\alpha_i] \in H^*(X)$ and $[\beta] \in H^*(\overline{\mathcal{M}}_{g,n})$, then the genus g and n-marked Gromov-Witten invariant of (X,ω) is defined as the integral

$$GW_{g,n}([\alpha_1], \cdots, [\alpha_n], [\beta]) = \int_{\overline{\mathcal{M}}_{g,n}(X,A)} ev_1^*(\alpha_1) \wedge \cdots \wedge ev_n^*(\alpha_n) \wedge \pi^*(\beta), \tag{3.2}$$

which, by Stokes' Theorem, is independent of the choices of α_i and β and also independent of the generic choice of (J,ν) since $\overline{\mathcal{M}}_{g,n}(X,A)$ is a closed orbifold (The integral (3.2) is defined to be zero unless $\dim\overline{\mathcal{M}}_{g,n}(X,A) = \sum_{i=1}^n \deg\alpha_i + \deg\beta$). Therefore, $GW_{g,n}$ is an invariant of the deformation class of ω parameterized by the cohomology classes $[\alpha_i]$ and β. In some sense, we can regard the Gromov-Witten invariants of X as counting with sign the number of (J,ν)-holomorphic maps from a n-marked Riemann surface of fixed genus g to X representing the second homology class A and intersecting some fixed generic representative cycles of $PD([\alpha_i])$. And we use the class $[\beta]$ to fix the conformal structure on the domain curve or the relative position of marked points.

[3]In general, the transversality is not always satisfied, the compactification of moduli space maybe have larger dimension than expected by index theorem. However, the compactification will carry a *virtual cycle* or *virtual fundamental class* which has the right dimension and, together with the map $\pi \times \mathbf{ev}$, can be used to define the GW invariants for general symplectic manifold.

3.2 Relative Gromov-Witten invariants

Let us review the construction by Ionel-Parker of relative Gromov-Witten invariants for a symplectic manifold (X, ω) and a codimension 2 symplectic submanifold V. Such relative invariants can be used to describe how GW invariants behave under symplectic connect sums along V. A stable map into the connect sum can be regarded as a pair of stable maps into the two sides matching in the middle overlap region, so a sum formula requires a count of stable maps in X. Curves in X in general position will intersect the submanifold V in a finite collection of points. Then relative invariants will still be a count of these curves in X and will keep track of how the curves intersect V.

To ensure that we can extend J and ν to the connect sum we require to take some special and so-called V-*compatible* pair (J, ν). Such a pair (J, ν) is no longer generic. Note that ω and J define a metric on X. Denote the orthogonal projection onto the normal bundle N_V of V by $\xi \mapsto \xi^N$.

Definition 3.1 *We say the pair (J, ν) is V-compatible if its 1-jet along V satisfies the following three conditions:*
(1) J preserves TV and $\nu^N|_V = 0$;
for all $\xi \in N_V$, $v \in TV$ and $w \in T\Sigma$
(2) $[(\nabla_\xi J + J\nabla_{J\xi} J)(v)]^N = [(\nabla_v J)\xi + J(\nabla_{Jv} J)\xi]^N$;
(3) $[(\nabla_\xi \nu + J\nabla_{J\xi} \nu)(w)]^N = [(J\nabla_{\nu(w)} J)\xi]^N$,
where ∇ is the pullback connection on u^*TX.

We denote by $\mathcal{J}^V$ the set of all V-compatible pairs $(J, \nu) \in \mathcal{J}$. The condition (1) in the Definition 3.1 ensures that V is a J-holomorphic submanifold, so (J, ν)-holomorphic curves in V are also (J, ν)-holomorphic in X. From the Lemma 3.3 in [IP1] we know that conditions (2) and (3) ensure that for each (J, ν)-holomorphic map u whose image lies in V, the operator $D_u^N : \Gamma(u^*N_V) \to \Omega^{0,1}(u^*N_V)$, by restricting the linearization of $u = \bar{\partial}_{(J,\nu)}$ to the normal bundle, is a complex linear operator.

Since the special pair (J, ν) is not generic, the moduli space for such pair may have higher dimension. For instance, there exist such curves having some entire components completely lying in V which are badly nontransverse to V. We will exclude such maps and only choose V-*regular* maps to define the relative invariant.

Definition 3.2 *We say a stable (J, ν)-holomorphic map into X is V-regular if no component and none of special points of its domain is mapped entirely into V.*

The V-regular maps (maybe with nodal domain) form an open subset of the set of stable maps, we denote the moduli space of V-regular maps by $\mathcal{M}^V_{g,n}(X, A)$. Ionel-Parker showed that for each V-regular map u, the inverse image $u^{-1}(V)$ consists of isolated points p_i on the domain distinct from the special points. Furthermore, each p_i has a well-defined multiplicity s_i equal to the order of contact of the image of u with V at p_i. We suppose there are altogether ℓ intersection points. Then we denote all multiplicities by a vector $s = (s_1, \cdots, s_\ell)$ with each integer $s_i \geq 1$. We define the degree, length and order of s by

$$\deg(s) = \sum s_i, \qquad \ell(s) = \ell, \qquad |s| = s_1 s_2 \cdots s_\ell.$$

Each vector s labels one component of $\mathcal{M}^V_{g,n}(X, A)$ with $\deg(s) = A \cdot V$:

$$\mathcal{M}^V_{g,n,s}(X, A) \subset \mathcal{M}_{g,n+\ell(s)}(X, A).$$

From the Lemma 4.2 in [IP1] we know that for generic compatible pair (J, ν), the irreducible part of $\mathcal{M}^V_{g,n,s}(X, A)$ is a manifold with

$$\dim \mathcal{M}^V_{g,n,s}(X, A) = 2c_1(A) + (\dim X - 6)(1 - g) + 2(n + \ell(s) - \deg(s)). \tag{3.3}$$

The next step is to construct a space that records the points where a V-regular map intersects V and records the homology class of the map. In order to define a relative invariant useful for a connect sum gluing theorem, we should record the homology class of the curve in $X \setminus V$ rather than in X. According to [IP1], two elements of $H_2(X \setminus V)$ represent the same element of $H_2(X)$ if they differ by an element of the set of rim tori $\mathcal{R} \subset H_2(X \setminus V)$. So the suitable homology-intersection data form a covering space $\mathcal{H}^V_X$ of $H_2(X) \times V^\ell$ with fiber $\mathcal{R}$.

To obtain a "relative virtual class" that enable us to define the relative GW invariant, we construct a compactification of each component of V-regular moduli space. Since we can think that there are $n + \ell(s)$ marked points on the domain, we take the closure of $\mathcal{M}^V_{g,n,s}(X, A)$ in the space of stable maps $\overline{\mathcal{M}}_{g,n+\ell(s)}(X, A)$, denote it by $C\mathcal{M}^V_{g,n,s}(X, A)$. Actually, the closure lies in the subset of $\overline{\mathcal{M}}_{g,n+\ell(s)}(X, A)$ consisting of stable maps whose last $\ell(s)$ marked points are mapped into V with associated multiplicities s_i.

Ionel-Parker analysed the structure of the closure $C\mathcal{M}^V_{g,n,s}(X, A)$ and showed that the closure contains strata corresponding to three different fundamental types of limits:

i). stable maps with no components or special points lying entirely in V;

ii). a stable with smooth domain which is mapped entirely into V;

iii). maps with some components in V and some off V.

From the Proposition 6.1 in [IP1] we know that for generic pair $(J, \nu) \in \mathcal{J}^V$, each stratum of the irreducible part of

$$C\mathcal{M}^V_{g,n,s}(X, A) \setminus \mathcal{M}^V_{g,n,s}(X, A)$$

is a manifold of dimension at least two less than the dimension of $\mathcal{M}^V_{g,n,s}(X, A)$.

To prove their Proposition 6.1, Ionel-Parker study the sequences u_n of V-regular maps by an iterated renormalization procedure and show that each such sequence limits to a stable map u with additional structure. The key point is that if some of components of such limit maps have images lying in V, then along each component in V there is a section ξ of the normal bundle of V satisfying an elliptic equation $D^N\xi = 0$. The ξ records the direction from which the image of that component came as it approached V. They call the stable maps with this additional structure V-*stable* maps, denoted by $\overline{\mathcal{M}}^V_{g,n,s}(X, A)$. Thus $\overline{\mathcal{M}}^V_{g,n,s}(X, A)$ compactifies the space of V-regular maps $\mathcal{M}^V_{g,n,s}(X, A)$ by adding frontier strata of codimension at least two.

Now we have a canonical map which can be regarded as the definition of relative Gromov-Witten invariants similar to the way that the GW invariants are defined in (3.1)

$$\varepsilon_V : \overline{\mathcal{M}}^V_{g,n,s}(X, A) \longrightarrow \overline{\mathcal{M}}_{g,n+\ell(s)} \times X^n \times \mathcal{H}^V_X, \tag{3.4}$$

where the last factor controls how the images of the maps intersect V. Then Theorem 8.1 in [IP1] showed that for generic $(J, \nu) \in \mathcal{J}^V$, the image of $\overline{\mathcal{M}}^V_{g,n,s}(X, A)$ under ε_V defines a homology class

$$GW^V_{X,A,g,n,s} \in H_*(\overline{\mathcal{M}}_{g,n+\ell(s)} \times X^n \times \mathcal{H}^V_X; \mathbb{Q})$$

of dimension as expressed in (5.1). Also this homology class is independent of the generic choice of the pair $(J,\nu)\in\mathcal{J}^V$ and depends only on the symplectic deformation equivalence class of (X,V,ω). Here we say (X,V,ω) is deformation equivalent to (X',V',ω') if there is a diffeomorphism $\phi: X'\to X$ such that $(X',\phi^{-1}(V),\omega)$ is isotopic to (X',V',ω').

Geometrically, relative GW invariants count with sign the number of V-stable maps with constraints on the complex structure of the domain, the images of marked points, and the way of the intersection with V. To see that, we can evaluate the homology class $GW^V_{X,A,g,n,s}$ on dual cohomology class. Choose $\kappa\in H^*(\overline{\mathcal{M}}_{g,n+\ell(s)})$, a vector $\alpha=(\alpha_1,\cdots,\alpha_n)$ of classes in $H^*(X)$, and $\gamma\in H^*(\mathcal{H}^V_X)$ satisfying

$$\dim\,\mathcal{M}^V_{g,n,s}(X,A)=\deg\,\kappa+\deg\,\alpha+\deg\,\gamma.$$

Then the evaluation pairing reexpresses the invariant as a collection of numbers

$$GW^V_{X,A,g,n,s}(\kappa,\alpha,\gamma)=\langle[GW^V_{X,A,g,n,s}],\kappa\cup\alpha\cup\gamma\rangle.$$

3.3 Open GW invariants for (X,ω,ϕ) of lower dimensions

Let us first introduce some notations used in [So]. We denote by (X,ω) a symplectic manifold of dimension $2n$ and by $L\subset X$ a Lagrangian submanifold. Let $\mathcal{J}_\omega$ denote the set of ω-tame almost complex structures on X, and let $J\in\mathcal{J}_\omega$. Let $\mathcal{P}$ denote the set of J-anti-linear inhomogeneous perturbation terms generalizing those introduced by Ruan and Tian in [RT1], and let $\nu\in\mathcal{P}$. Denote a Riemann surface with boundary by $(\Sigma,\partial\Sigma)$, we will fix a conformal structure j on Σ through this paper. Suppose $\partial\Sigma=\coprod_{a=1}^m(\partial\Sigma)_a$, where each $(\partial\Sigma)_a\simeq S^1$. Let

$$\mathbf{d}=(d,d_1,\cdots,d_m)\in H_2(X,L)\oplus H_1(L)^{\oplus m},$$

and $\mathbf{k}=(k_1,\cdots,k_m)\in\mathbb{N}^m\cup\{0,\cdots,0\}$, $l\in\mathbb{N}\cup\{0\}$. Then we denote by

$$\mathcal{M}_{\mathbf{k},l}(\Sigma,L,\mathbf{d})$$

the moduli space of (J,ν)-holomorphic maps $u:(\Sigma,\partial\Sigma)\to(X,L)$ with k_a marked points $z_1,\cdots,z_{k_a}$ on $(\partial\Sigma)_a$ and l marked points $w_1,\cdots,w_l$ on Σ such that $u_*([\Sigma,\partial\Sigma])=d$ and $u|_{(\partial\Sigma)_a*}([(\partial\Sigma)_a])=d_a$. Its Gromov compactification is denoted by $\overline{\mathcal{M}}_{\mathbf{k},l}(\Sigma,L,\mathbf{d})$. We have the evaluation maps

$$evb_{ai}:\overline{\mathcal{M}}_{\mathbf{k},l}(\Sigma,L,\mathbf{d})\to L,\quad i=1,\cdots,k_a,\ a=1,\cdots,m,$$

$$evi_j:\overline{\mathcal{M}}_{\mathbf{k},l}(\Sigma,L,\mathbf{d})\to X,\quad j=1,\cdots,l.$$

In fact, the moduli space above can be considered as the zero locus of a Fredholm section of a Banach bundle. We denote by $B^{1,p}(\Sigma,L,\mathbf{d})$ the Banach manifold of $W^{1,p}$ maps $u:(\Sigma,\partial\Sigma)\to(X,L)$ such that $u_*([\Sigma,\partial\Sigma])=d$ and $u|_{(\partial\Sigma)_a*}([(\partial\Sigma)_a])=d_a$. And define

$$B^{1,p}_{\mathbf{k},l}(\Sigma,L,\mathbf{d}):=B^{1,p}(\Sigma,L,\mathbf{d})\times\prod_a(\partial\Sigma)_a^{k_a}\times\Sigma^l\setminus\triangle,$$

where $\triangle$ denotes the subset of the product in which two marked points coincide. Elements of $B^{1,p}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})$ are denoted by $\mathbf{u} = (u, \vec{z}, \vec{w})$, where $\vec{z} = (z_{ai})$ and $\vec{(w)} = (w_j)$. The evaluation maps can also be similarly defined on this larger space

$$evb_{ai} : B^{1,p}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) \to L, \quad i = 1, \cdots, k_a,\ a = 1, \cdots, m,$$

$$evi_j : B^{1,p}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) \to X, \quad j = 1, \cdots, l,$$

$$evb_{ai}(\mathbf{u}) = u(z_{ai}), \quad evi_j(\mathbf{u}) = u(w_j).$$

The total evaluation map is denoted by

$$\mathbf{ev} := \prod_{a,i} evb_{ai} \times \prod_j evi_j : B^{1,p}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) \to L^{|\mathbf{k}|} \times X^l.$$

Then the Banach space bundle $\mathcal{E} \to B^{1,p}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})$ is defined fiberwise with

$$\mathcal{E}_{\mathbf{u}} := L^p(\Sigma, \Omega^{0,1}(u^*TX)).$$

We define the section of this bundle as

$$\bar{\partial}_{(J,\nu)} : B^{1,p}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) \to \mathcal{E}$$

which is the ν-perturbed Cauchy-Riemann operator (see section 4 in [So]). The vertical component of the linearization of this section is denoted by

$$D := D\bar{\partial}_{(J,\nu)} : TB^{1,p}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) \to \mathcal{E}.$$

The parameterized moduli space is the zero locus of the section $\bar{\partial}_{(J,\nu)}$ which is denoted by

$$\widetilde{\mathcal{M}}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) := \bar{\partial}^{-1}_{J,\nu}(0) \subset B^{1,p}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}).$$

We follow [So] to define our moduli space $\mathcal{M}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) \subset \widetilde{\mathcal{M}}^*_{\mathbf{k},l}(\Sigma, L, \mathbf{d})$ as an appropriate section (or say slice) of the reparameterization group action, we refer the reader to the section 4 in [So] for details of construction.

Then let

$$\mathcal{L} := \det(D) \to B^{1,p}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})$$

be the determinant line bundle of the family of Fredholm operators D.

According to the Proposition 3.1 in [So], if the Lagrangian submanifold L admits a (relative) $Pin^{\pm}$ structure, then there exists a canonical isomorphism of line bundles

$$\mathcal{L} \xrightarrow{\sim} \bigotimes_{a,i} evb^*_{ai} \det(TL).$$

Let us then suppose there exists an anti-symplectic involution ϕ such that $L = \mathrm{Fix}(\phi)$. And suppose Σ is biholomorphic to its conjugation $\bar{\Sigma}$, *i.e.* there exists an anti-holomorphic involution $c : \Sigma \to \Sigma$. Define

$$\mathcal{J}_{\omega,\phi} := \{J \in \mathcal{J}_\omega |\ \phi^* J = -J\}.$$

Fix $J \in \mathcal{J}_{\omega,\phi}$, define $\mathcal{P}_{\phi,c}$ to be the set of $\nu \in \mathcal{P}$ satisfying $d\phi \circ \nu \circ dc = \nu$, then take $\nu \in \mathcal{P}_{\phi,c}$. Thus, from the (J,ν)-holomorphic map $u : (\Sigma, \partial\Sigma) \mapsto (X, L)$ we can define its conjugate (J,ν)-holomorphic map $\tilde{u} = \phi \circ u \circ c$ representing the homology class $\tilde{d} = [\tilde{u}]$. So we have an induced map

$$\phi' : B^{1,p}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) \to B^{1,p}_{\mathbf{k},l}(\Sigma, L, \tilde{\mathbf{d}})$$

given by

$$\mathbf{u} = (u, \vec{z}, \vec{w}) \mapsto \tilde{\mathbf{u}} = (\tilde{u},\ (c|_{\partial\Sigma})^{|\mathbf{k}|}\vec{z},\ c^l\vec{w}).$$

We denote the relevant Banach space bundle by $\widetilde{\mathcal{E}} \to B^{1,p}_{\mathbf{k},l}(\Sigma, L, \tilde{\mathbf{d}})$ with fiber

$$\widetilde{\mathcal{E}}_{\tilde{\mathbf{u}}} := L^p(\Sigma, \Omega^{0,1}(\tilde{u}^*TX)).$$

And we can similarly get a determinant line bundle of a family of Fredholm operators $\widetilde{D} : TB^{1,p}_{\mathbf{k},l}(\Sigma, L, \tilde{\mathbf{d}}) \to \widetilde{\mathcal{E}}$ as

$$\widetilde{\mathcal{L}} := \det(\widetilde{D}) \to B^{1,p}_{\mathbf{k},l}(\Sigma, L, \tilde{\mathbf{d}}).$$

The evaluation maps can also be similarly defined as

$$\widetilde{ev}b_{ai} : B^{1,p}_{\mathbf{k},l}(\Sigma, L, \tilde{\mathbf{d}}) \to L, \quad i = 1, \cdots, k_a,\ a = 1, \cdots, m,$$

$$\widetilde{ev}i_j : B^{1,p}_{\mathbf{k},l}(\Sigma, L, \tilde{\mathbf{d}}) \to X, \quad j = 1, \cdots, l,$$

$$\widetilde{ev}b_{ai}(\tilde{\mathbf{u}}) = \tilde{u}(c(z_{ai})), \quad \widetilde{ev}i_j(\tilde{\mathbf{u}}) = \tilde{u}(c(w_j)).$$

The total evaluation map is denoted by

$$\widetilde{\mathbf{ev}} := \prod_{a,i} \widetilde{ev}b_{ai} \times \prod_j \widetilde{ev}i_j : B^{1,p}_{\mathbf{k},l}(\Sigma, L, \tilde{\mathbf{d}}) \to L^{|\mathbf{k}|} \times X^l.$$

Moreover, we can define a map

$$\Phi : \mathcal{E} \to \widetilde{\mathcal{E}}$$

$$\xi \mapsto d\phi \circ \xi \circ dc$$

covering ϕ'. Also Φ induces a map $\Psi : \mathcal{L} \to \widetilde{\mathcal{L}}$ covering ϕ'. We still follow [So] denote

$$\mathcal{L}' := \mathrm{Hom}(\bigotimes_{a,i} evb^*_{ai}\det(TL), \mathcal{L}) \simeq \mathcal{L} \otimes \bigotimes_{a,i} evb^*_{ai}\det(TL)^*$$

$$\widetilde{\mathcal{L}}' := \mathrm{Hom}(\bigotimes_{a,i} \widetilde{ev}b^*_{ai}\det(TL), \widetilde{\mathcal{L}}) \simeq \widetilde{\mathcal{L}} \otimes \bigotimes_{a,i} \widetilde{ev}b^*_{ai}\det(TL)^*.$$

Thus, Ψ also induces a map Ψ' between $\mathcal{L}'$ and $\widetilde{\mathcal{L}}'$ covering ϕ'.

From the Proposition 3.1 and Definition 3.2 in [So] we know that both $\mathcal{L}'$ and $\widetilde{\mathcal{L}}'$ have canonical orientation. So the map Ψ' may either preserve the orientation component or reverse the orientation of some connected components. We say the sign of Ψ' is 0 if Ψ' preserves the orientation of $\mathcal{L}'$ to that of $\widetilde{\mathcal{L}}'$, otherwise, we say the sign of Ψ' is 1.

Now we have two moduli spaces $\mathcal{M}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})$ and, corresponding to the anti-symplectic involution ϕ, $\mathcal{M}_{\mathbf{k},l}(\Sigma, L, \tilde{\mathbf{d}})$. And we can restrict the two total evaluation maps

$$\mathbf{ev} : \mathcal{M}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}) \to L^{|\mathbf{k}|} \times X^l,$$

$$\widetilde{\mathbf{ev}} : \mathcal{M}_{\mathbf{k},l}(\Sigma, L, \tilde{\mathbf{d}}) \to L^{|\mathbf{k}|} \times X^l.$$

For generic choice of points $\vec{x} = (x_{ai})$, $x_{ai} \in L$, and pairs of points $\vec{y}_+ = (y_j^+)$, $\vec{y}_- = (y_j^-)$ such that $y_j^+ = \phi(y_j^-)$, $j = 1, \cdots, l$, the two total evaluation maps will be transverse to

$$\prod_{a,i} x_{ai} \times \prod_j y_j^+ \quad \text{and} \quad \prod_{a,i} x_{ai} \times \prod_j y_j^-$$

in $L^{|\mathbf{k}|} \times X^l$, respectively. The index theorem for D implies the following dimension condition

$$(n-1)(|\mathbf{k}| + 2l) = n(1-g) + \mu(d) - \dim\, Aut(\Sigma), \tag{3.5}$$

where $\mu : H_2(X, L) \to \mathbb{Z}$ denote the Maslov index, g denote the genus of the closed Riemann surface $\Sigma \cup_{\partial\Sigma} \bar{\Sigma}$ obtained by doubling Σ. Note that $\mu(d) = \mu(\tilde{d})$, we can define a number as

$$M(\mathbf{d}, \phi, \mathbf{k}, l) = \#\mathbf{ev}^{-1}(\vec{x}, \vec{y}_+) + \#\widetilde{\mathbf{ev}}^{-1}(\vec{x}, \vec{y}_-), \tag{3.6}$$

where $\#$ denotes the signed count with the sign of a given point, for example $\mathbf{u} \in \mathbf{ev}^{-1}(\vec{x}, \vec{y}_+)$, depending on whether or not the isomorphism

$$d\mathbf{ev}_{\mathbf{u}} : \det(T\mathcal{M}_{\mathbf{k},l}(\Sigma, L, \mathbf{d}))_{\mathbf{u}} \xrightarrow{\sim} \mathbf{ev}^* \det(T(L^{|\mathbf{k}|} \times X^l))_{\mathbf{u}}$$

agrees with the underlying canonical isomorphism appearing in the Theorem 1.1 in [So]

$$\det(T\mathcal{M}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})) \xrightarrow{\sim} \bigotimes_{a,i} evb_{ai}^* \det(TL).$$

In particular, if $d = \tilde{d}$ we just define the same number as the one defined by Solomon

$$M(\mathbf{d}, \phi, \mathbf{k}, l) = N_{\Sigma,\mathbf{d},\mathbf{k},l} = \#\mathbf{ev}^{-1}(\vec{x}, \vec{y}),$$

where $(\vec{x}, \vec{y})$ is a real configuration, *i.e.* $l = 2c$, $\vec{y} = \{y_1^+, \cdots, y_c^+, y_1^-, \cdots, y_c^-\}$ satisfying $y_i^+ = \phi(y_i^-)$, $i = 1, \cdots, c$. For such special $d \in H_2(X, L)$, Solomon [So] proved that $N_{\Sigma,\mathbf{d},\mathbf{k},l}$ are invariants if L is orientable and $\dim L = 3$ or if L may not orientable and $\dim L = 2$. However, for general homology class d, we might not expect that $M(\mathbf{d}, \phi, \mathbf{k}, l)$ is invariant.

In order to define enumerative invariant not only for restricted homology class d, let us introduce more necessary notations. We denote by $\mathbb{d} = d_{\mathbb{C}}$ the doubling of d. For any homology class $\beta \in H_2(X, L)$, denote

$$\bar{\beta} = (\beta, \beta_1, \cdots, \beta_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m},$$

and we denote the set of $(\mathbf{k}, l)$-real configurations by

$$\mathcal{R}(\vec{x}, \vec{y}) = \{\vec{r} = (\vec{x}, \vec{\xi}\,) = (x_{11}, \cdots, x_{mk_m}, \xi_1, \cdots, \xi_l) | \xi_j = y_j^+ \text{ or } \xi_j = y_j^-, j = 1, \cdots, l\}.$$

Moreover, denote by $\mathbf{ev}_{(\beta,\vec{r})}$ the evaluation map

$$\mathbf{ev}_{(\beta,\vec{r})} : \mathcal{M}_{\mathbf{k},l}(\Sigma, L, \bar{\beta}) \to L^{|\mathbf{k}|} \times X^l,$$

$$(u, \vec{z}, \vec{w}) \mapsto (\vec{x} = u(\vec{z}), \vec{\xi} = u(\vec{w})).$$

Now, we can define one number

$$I(\mathbb{d}, \phi, \mathbf{k}, l) := \sum_{\forall \beta:\beta_{\mathbb{C}}=\mathbb{d}; \forall \vec{r}\in\mathcal{R}(\vec{x},\vec{y})} \#\mathbf{ev}^{-1}_{(\beta,\vec{r})}(\vec{x}, \vec{\xi}\,) \tag{3.7}$$

In the sequel, we will show that the number $I(\mathbb{d}, \phi, \mathbf{k}, l)$ is an invariant, provided L is orientable and $\dim L = 3$.

Remark. The definition of invariants (3.7) is a generalization of the one by Cho [C] to higher genus case.

Before proving the invariance, we rewrite the number $I(\mathbb{d}, \phi, \mathbf{k}, l)$ as a sum of some integrals. Let $\Omega^*(L, \det(TL))$ denote differential forms on L with values in $\det(TL)$, and let $\Omega^*(X)$ denote ordinary differential forms on X. Let $\alpha_{ai} \in \Omega^n(L, \det(TL))$, $a = 1, \cdots, m; i = 1, \cdots, k_a$, represent the Poincaré dual of the point x_{ai} in $H^n(L, \det(TL))$, which is the cohomology of L with coefficients in the flat line bundle $\det(TL)$. And let $\gamma_j \in \Omega^{2n}(X)$ represent the Poincaré dual of ξ_j for $j = 1, \cdots, l$. Then it is obvious that we have the following expression

$$I(\mathbb{d}, \phi, \mathbf{k}, l) = \sum_{\forall \beta:\beta_{\mathbb{C}}=\mathbb{d}} \sum_{\substack{\gamma_j \,:\, [\gamma_j] = \mathrm{PD}(\xi_j), \\ \forall (\vec{x}, \vec{\xi}) \,\in\, \mathcal{R}(\vec{x}, \vec{y})}} \int_{\mathcal{M}_{k,l}(\Sigma, L, \bar{\beta})} evb^*_{11}(\alpha_{11}) \wedge \cdots \wedge evi^*_l \gamma_l \tag{3.8}$$

To show that the definition (3.7) is independent of the choices of points $\vec{x}$ and pairs of points $(\vec{y}_+, \vec{y}_-)$ is equivalent to proving the expression (3.8) is independent of the choices of $\det(TL)$-valued n-forms α_{ai} and pairs of $2n$-forms (γ^+_j, γ^-_j), where $\gamma^{\pm}_j$ represent the Poincaré dual of $y^{\pm}_j$ for $j = 1, \cdots, l$.

• *Proof of invariance*

Here, for simply showing the main idea of the method, we only give a sketchy proof of the invariance for a special case, with respect to general homology class $d \in H_2(X)$. A complete proof for the special homology class $d = \tilde{d}$ is given in [So]. Assume that $\dim L = 3$ and L is orientable. We will show that the definition of $I(\mathbb{d}, \phi, \mathbf{k}, l)$ does not depend on the choice of points $\vec{x} = (x_{ai})$, $x_{ai} \in L$, and pairs of points $\vec{y}_+ = (y^+_j)$, $\vec{y}_- = (y^-_j)$. Suppose that we are given different points $\vec{x'}$ and $\vec{y'}_{\pm}$ satisfying the same generic conditions.

Let us denote

$$\mathbf{x} : [0, 1] \to L^{|k|}, \quad \mathbf{x}(0) = \vec{x}, \quad \mathbf{x}(1) = \vec{x'},$$

$$\mathbf{y}^{\pm} : [0, 1] \to X^l, \quad \mathbf{y}^+(t) = \phi(\mathbf{y}^-(t)),$$

$$\mathbf{y}^{\pm}(0) = \vec{y}_{\pm}, \quad \mathbf{y}^{\pm}(1) = \vec{y'}_{\pm},$$

$$\Xi:[0,1]\to X^l,\quad \Xi(0)=\vec{\xi},\quad \Xi(1)=\vec{\xi'},$$

moreover, we require that $\xi_j=y_j^+$ (or y_j^-) if and only if $\xi'_j=y_j'^+$ (or $y_j'^-$). Denote the set of all paths by

$$\mathbf{R}:=\mathbf{R}(\mathbf{x},\mathbf{y}^\pm)=\{(\mathbf{x},\Xi)\}$$

And denote

$$\mathcal{W}(\mathbf{x},\Xi,\bar{\beta}):=\mathcal{M}_{\mathbf{k},l}(\Sigma,L,\bar{\beta})_{\mathbf{ev}_{(\beta,\vec{r})}}\times_{(\mathbf{x}\times\Xi)\circ\triangle}([0,1]),$$

$$\mathcal{W}:=\mathcal{W}(\mathbf{x},\mathbf{y}^\pm,\alpha):=\bigcup_{\substack{\beta:\beta_{\mathbb{C}}=\mathbb{d},\\(\mathbf{x},\Xi)\in\mathbf{R}}}\mathcal{W}(\mathbf{x},\Xi,\bar{\beta}).$$

Note that $\mathcal{W}$ gives a smooth oriented cobordism between

$$\bigcup_{\forall\beta:\beta_{\mathbb{C}}=\mathbb{d};\forall\vec{r}\in\mathcal{R}(\vec{x},\vec{y})}\mathbf{ev}^{-1}_{(\beta,\vec{r})}(\vec{x},\vec{\xi}\,)\quad\text{ and }\quad\bigcup_{\forall\beta:\beta_{\mathbb{C}}=\mathbb{d};\forall\vec{r'}\in\mathcal{R}(\vec{x'},\vec{y'})}\mathbf{ev}^{-1}_{(\beta,\vec{r'})}(\vec{x'},\vec{\xi'}\,).$$

Since in general $\mathcal{W}$ is noncompact, in order to prove the invariance of $I(\mathbb{d},\phi,\mathbf{k},l)$, we must research the stable boundary $\partial_G\mathcal{W}$ arising from the Gromov compactification of $\mathcal{W}$.

We still adopt the notations in [So]. Denote by

$$B^{\#}=B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(\Sigma,L,\bar{\beta}',\beta''):=B^{1,p}_{\mathbf{k}'+e_b,l'}(\Sigma,L,\bar{\beta}')_{evb'_0}\times_{evb''_0}B^{1,p}_{\mathbf{k}''+1,l''}(D^2,L,\beta'')$$

the space of $W^{1,p}$ stable maps with only one disc bubbling off the boundary component $(\partial\Sigma)_b$ along with k'' of the marked points on $(\partial\Sigma)_b$ and l'' of the interior marked points, representing the class $\beta''\in H_2(X,L)$, where $\bar{\beta}'=(\beta',\beta_1,\cdots,\beta'_b,\cdots,\beta_m)\in H_2(X,L)\oplus H_1(L)^{\oplus m}$, satisfying $\beta'+\beta''=\beta$, $\beta'_b+\partial\beta''=\beta_b$, and

$$\mathbf{k}'+e_b=(k_1,\cdots,k_b+1,\cdots,k_m),$$

$\sigma\subset[1,k_b]$, $\varrho\subset[1,l]$ denote the subsets of boundary and interior bubble off marked points, respectively.

We write the element $\mathbf{u}\in B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(\Sigma,L,\bar{\beta}',\beta'')$ as $\mathbf{u}=(\mathbf{u}',\mathbf{u}'')$, where

$$\mathbf{u}'\in B^{1,p}_{\mathbf{k}'+e_b,l'}(\Sigma,L,\bar{\beta}'),\qquad \mathbf{u}''\in B^{1,p}_{\mathbf{k}''+1,l''}(D^2,L,\beta'')$$

such that

$$evb'_0(\mathbf{u}')=evb''_0(\mathbf{u}'').$$

Similarly, we have the Banach space bundle $\mathcal{E}^{\#}\to B^{\#}$, the section $\bar{\partial}^{\#}_{(J,\nu)}:B^{\#}\to\mathcal{E}^{\#}$, the operator $D^{\#}:TB^{\#}\to\mathcal{E}^{\#}$, the moduli space $\mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(\Sigma,L,\bar{\beta}',\beta'')$, the determinant line bundle $\mathcal{L}^{\#}:=\det(D^{\#})$ and

$$\mathcal{L}^{\#'}=\mathcal{L}^{\#}\otimes\bigotimes_{a,i}evb^*_{ai}\det(TL)^*.$$

Since there exists a standard conjugation $c:D^2\to D^2$, the anti-symplectic involution ϕ on X induces a map

$$\phi^{\#}:B^{\#}=B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(\Sigma,L,\bar{\beta}',\beta'')\to B^{1,p}_{\mathbf{k},\sigma,l,\varrho}(\Sigma,L,\bar{\beta}',\widetilde{\beta''})=\widetilde{B}^{\#}.$$

$$\mathbf{u} = (\mathbf{u}', \mathbf{u}'') = [\mathbf{u}', (u'', \vec{z}'', \vec{w}'')] \mapsto \tilde{\mathbf{u}} = (\mathbf{u}', \tilde{\mathbf{u}}'') = [\mathbf{u}', (\tilde{u}'', \ (c|_{\partial\Sigma})^{k''+1}\vec{z}'', \ c^{l''}\vec{w}'')].$$

Similarly, we have maps

$$\Phi^{\#} : \mathcal{E}^{\#} \to \widetilde{\mathcal{E}}^{\#}, \quad \Psi^{\#} : \mathcal{L}^{\#} \to \widetilde{\mathcal{L}}^{\#}, \quad \Psi^{\#'} : \mathcal{L}^{\#'} \to \widetilde{\mathcal{L}}^{\#'}$$

covering $\phi^{\#}$.

We can generically choose $(\mathbf{x}, \mathbf{y}^{\pm})$ such that $(\mathbf{x}, \Xi) = [\mathbf{x}, (\Xi', \Xi'')] \in \mathbf{R}$ is transverse to the total evaluation map

$$\mathbf{ev}_{(\beta'+\beta'', \vec{r})} : \mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(\Sigma, L, \bar{\beta}', \beta'') \to L^{|\mathbf{k}|} \times X^{l}, .$$

We denote by

$$\partial_G \mathcal{W}_{\sigma,\varrho,\beta=\beta'+\beta''}(\Xi'') := \mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(\Sigma, L, \bar{\beta}', \beta'')_{\mathbf{ev}_{(\beta'+\beta'',\vec{r})}} \times_{(\mathbf{x}\times(\Xi',\Xi''))\circ\triangle} [0,1],$$

the boundary stratum of the cobordism $\mathcal{W}$ arising from Gromov compactification. And

$$\partial_G \mathcal{W}_{\sigma,\varrho, \ \beta',\beta''} = \partial_G \mathcal{W}_{\sigma,\varrho, \ \beta'+\beta''}(\Xi'') \bigcup \partial_G \mathcal{W}_{\sigma,\varrho, \ \beta'+\widetilde{\beta''}}(\phi(\Xi'')),$$

$$\partial_G \mathcal{W} = \bigcup_{\substack{a \in [1,m], \sigma \subset [1,k_a], \\ \varrho \subset [1,l]}} \ \bigcup_{\beta',\beta'' : \ (\beta'+\beta'')_{\mathbb{C}} = \mathbb{d}} \partial_G \mathcal{W}_{\sigma,\varrho, \ \beta',\beta''} \ .$$

So we have an induced involution map on the boundary of cobordism

$$\phi_{\partial} : \partial_G \mathcal{W} \to \partial_G \mathcal{W},$$

$$\partial_G \mathcal{W}_{\sigma,\varrho, \ \beta'+\beta''}(\Xi'') \to \partial_G \mathcal{W}_{\sigma,\varrho, \ \beta'+\widetilde{\beta''}}(\phi(\Xi'')).$$

Using the similar arguments as in [So], we can show that this map is fixed point free. We below will show that it is orientation reversing if L is orientable and $\dim L = 3$.

Recall that, under our assumptions, the formula (19) in [So] expressing the sign of $\Psi^{\#'}$ still works. And it can be simplified as

$$sign(\Psi^{\#'}) = \frac{\mu(\beta'')}{2} + k'' + 1 + l'' \qquad \mod 2, \tag{3.9}$$

since $w_2 = 0$ as L admits $Pin^{\pm}$ structure. And similar to Solomon's arguments, the map

$$\phi^{\#} : \mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(\Sigma, L, \bar{\beta}', \beta'') \to \mathcal{M}_{\mathbf{k},\sigma,l,\varrho}(\Sigma, L, \bar{\beta}', \widetilde{\beta''})$$

has the same sign. Since $\dim L = 3$, we see that the involution ϕ acts on X reversing the orientation since $\phi^*\omega^3 = -\omega^3$. So ϕ acts non-trivially on l'' of the factors of X^l. Therefore, the sign of the map between the two fiber products

$$\phi_{\partial} : \partial_G \mathcal{W}_{\sigma,\varrho, \ \beta'+\beta''}(\Xi'') \to \partial_G \mathcal{W}_{\sigma,\varrho, \ \beta'+\widetilde{\beta''}}(\phi(\Xi''))$$

should be independent of l''.

Note that each set $\partial_G\mathcal{W}_{\sigma,\varrho,\beta',\beta''}$ is nonempty if and only if the following dimension condition is satisfied

$$\mu(\beta'') = 2k'' + 4l''. \tag{3.10}$$

Thus we have

$$sign(\phi_\partial) = \frac{\mu(\beta'')}{2} + k'' + 1 \cong 1 \mod 2.$$

That is to say, the map ϕ_∂ reverses orientation. That means $\#\partial_G\mathcal{W} = 0$. Therefore, we have

$$\begin{aligned} 0 = \#\partial\mathcal{W} &= \textstyle\sum_{\beta;\vec{r}'} \#\mathbf{ev}^{-1}_{(\beta,\vec{r}')}(\vec{x}', \vec{\xi}'\,) - \sum_{\beta;\vec{r}} \#\mathbf{ev}^{-1}_{(\beta,\vec{r})}(\vec{x}, \vec{\xi}\,) + \#\partial_G\mathcal{W} \\ &= I'(\mathbb{d}, \phi, \mathbf{k}, l) - I(\mathbb{d}, \phi, \mathbf{k}, l). \end{aligned}$$

So integers $I(\mathbb{d}, \phi, \mathbf{k}, l)$ are independent of the choice of $(\vec{x}, \vec{y}_\pm)$. Equivalently, we can say that the integral in (3.8) is independent of the choices of α_{ai} and γ_j. Similarly, we can prove that $I(\mathbb{d}, \phi, \mathbf{k}, l)$ are independent of the generic choice of $J \in \mathcal{J}_{\omega,\phi}$, and the choice of inhomogeneous perturbation $\nu \in \mathcal{P}_{\phi,c}$. That means $I(\mathbb{d}, \phi, \mathbf{k}, l)$ are invariants of the triple (X, ω, ϕ).

4 V-compatible pair (J, ν)

We now come to extend Solomon's open symplectic invariants $N_{\Sigma,\mathbf{d},\mathbf{k},l}$ with $d = \tilde{d}$ (or in general our $I(\mathbb{d}, \phi, \mathbf{k}, l)$) expressed in subsection 3.3 to open invariants of (X, ω, ϕ) relative to a codimension 2 symplectic submanifold V such that $\phi|_V$ is also an anti-symplectic involution on $(V, \omega|_V)$. Recall the assumption that $L = \mathrm{Fix}(\phi) \neq \emptyset$ is a Lagrangian submanifold. Note that $V \cap L$ might be empty set. If $V \cap L = \mathrm{Fix}(\phi|_V) \neq \emptyset$, then we denote by $L_V = V \cap L$ which is a Lagrangian submanifold of $(V, \omega|_V)$. We consider open curves generically intersect V in a finite collection of points. Such relative open invariants will count these open curves satisfying some constraints. In particular, if $L \cap V = \emptyset$, we will not encounter extra codimension 1 boundary of moduli space except the moduli space of pseudoholomorphic maps with a bubble disc. If $\dim L \leq 3$, under some assumptions, we will define relatively open invariants for domain curves of any genus with fixed conformal structures.

Let us reset our notations for new discussion. Suppose $\dim X = 2n$. We denote still by $(\Sigma, \partial\Sigma)$ the domain Riemann surface with m boundary components and with fixed conformal structure j_Σ. That means we also don't deal with the case that the degenerations of Σ may occur. Denote by g the genus of the closed Riemann surface $\Sigma \cup_{\partial\Sigma} \bar{\Sigma}$ obtained by doubling Σ. If Σ is a closed Riemann surface then g is just the genus of itself. Assume that there are k_a distinct marked points $\{z_{a1}, \cdots, z_{ak_a}\}$ on the boundary component $(\partial\Sigma)_a$, $a = 1, \cdots, m$, and l distinct interior marked points $\{w_1, \cdots, w_l\}$. Denote by $\mathbf{k} = (k_1, \cdots, k_m)$. Additionally, we assume that there are $\mathbb{k}_a$ marked points $\{p_{a1}, \cdots, p_{a\mathbb{k}_a}\}$ on each boundary component $(\partial\Sigma)_a$, $a = 1, \cdots, m$, which are different from $\{z_{a1}, \cdots, z_{ak_a}\}$ and are mapped to the intersection points of V and the image of our open curve $u : (\Sigma, \partial\Sigma) \to (X, L)$ if $V \cap L \neq \emptyset$. Denote by $\mathbb{k} = (\mathbb{k}_1, \cdots, \mathbb{k}_m)$. Also, we assume there are $\mathbb{l}$ interior marked points $\{q_1, \cdots, q_{\mathbb{l}}\}$ which are different from $\{w_1, \cdots, w_l\}$ and are mapped to the intersection points of V and the image of u. For given homology class $\mathbf{d} = (d, d_1, \cdots, d_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m}$, and given a pair (J, ν), denoted

by $\mathcal{M}_{\mathbf{k},l,\Bbbk,\mathbb{l}}(X, L, \mathbf{d})$ the moduli space of $(\mathbf{k}, l, \Bbbk, \mathbb{l})$-marked (J, ν) pseudoholomorphic maps such that $u_*([\Sigma, \partial\Sigma]) = d$ and $u|_{(\partial\Sigma)_a *}([(\partial\Sigma)_a]) = d_a$, *i.e.* representing $\mathbf{d}$, which is the zero set of

$$\bar{\partial}_{(J,\nu)}\ u = \bar{\partial}_J u - \nu = \frac{1}{2}(du + J \circ du \circ j_\Sigma) - \nu. \tag{4.1}$$

We denote the ambient space, which is the Sobolev completion of $\mathcal{M}_{\mathbf{k},l,\Bbbk,\mathbb{l}}(X, L, \mathbf{d})$, by

$$B^{1,p}_{\mathbf{k},l,\Bbbk,\mathbb{l}}(X, L, \mathbf{d}) := B^{1,p}(X, L, \mathbf{d}) \times \prod_a (\partial\Sigma)_a^{(k_a + \Bbbk_a)} \times \Sigma^{(l+\mathbb{l})} \backslash \triangle,$$

where $B^{1,p}(X, L, \mathbf{d})$ denote the Banach manifold of $W^{1,p}$ maps $u : (\Sigma, \partial\Sigma) \to (X, L)$ representing $\mathbf{d}$. If we use vectors $\vec{z} = (z_{ai}), \vec{w} = (w_j), \vec{p} = (p_{ai}), \vec{q} = (q_j)$ to denote marked points respectively, then we denote elements of $B^{1,p}_{\mathbf{k},l,\Bbbk,\mathbb{l}}(X, L, \mathbf{d})$ by $\mathbf{u} = (u, \vec{z}, \vec{w}, \vec{p}, \vec{q})$. We simply denote the restricted pullback bundle $(u|_{\partial\Sigma})^* TL$ by u^*TL. By straightforward computations we have the following proposition.

Proposition 4.1 *The linearization of (4.1) at* $\mathbf{u} \in \mathcal{M}_{\mathbf{k},l,\Bbbk,\mathbb{l}}(X, L, \mathbf{d})$ *is*

$$D_{\mathbf{u}} := D_{\mathbf{u}} \bar{\partial}_{(J,\nu)} : \Gamma[(\Sigma, \partial\Sigma), (u^*TX, u^*TL)] \to \Omega^{0,1}(\Sigma, u^*TX)$$

$$\begin{aligned} D_{\mathbf{u}}(\xi) &= \frac{1}{2}[\nabla\xi + J \circ \nabla\xi \circ j + \nabla_\xi J \circ du \circ j] \\ &\quad + \frac{1}{2}[T(\xi, du) + JT(\xi, du \circ j)] - \nabla_\xi \nu \end{aligned} \tag{4.2}$$

where ∇ *is the pullbacked connection on* u^*TX *and* $T(\zeta, \eta) = \nabla_\zeta \eta - \nabla_\eta \zeta - [\zeta, \eta]$ *is the torsion of* ∇.

Denote by

$$\mathcal{D}_u(\xi) = \frac{1}{2}[\nabla\xi + J \circ \nabla\xi \circ j + \nabla_\xi J \circ du \circ j] - \nabla_\xi \nu. \tag{4.3}$$

We can similarly define the Banach space bundle $\mathcal{E} \to B^{1,p}_{\mathbf{k},l,\Bbbk,\mathbb{l}}(X, L, \mathbf{d})$ fiberwise by

$$\mathcal{E}_{\mathbf{u}} := L^p(\Sigma, \Omega^{0,1}(u^*TX)).$$

So we still think

$$\bar{\partial}_{(J,\nu)} : B^{1,p}_{\mathbf{k},l,\Bbbk,\mathbb{l}}(X, L, \mathbf{d}) \to \mathcal{E}$$

as a section of $\mathcal{E}$. And we also denote by

$$D := D\bar{\partial}_{(J,\nu)} : TB^{1,p}_{\mathbf{k},l,\Bbbk,\mathbb{l}}(X, L, \mathbf{d}) \to \mathcal{E}$$

the vertical component of the linearization of $\bar{\partial}_{(J,\nu)}$. From the Proposition 6.14 in [Liu] we know that for each $\mathbf{u} \in B^{1,p}_{\mathbf{k},l,\Bbbk,\mathbb{l}}(X, L, \mathbf{d})$, both $D_{\mathbf{u}}$ and $\mathcal{D}_u$ are Fredholm operators with the same index $\mathrm{ind}(D_{\mathbf{u}}) = \mathrm{ind}(\mathcal{D}_u) = \mu + n(1-g)$, where $\mu = \mu(u^*TX, u^*TL)$ is the total boundary Maslov index associated with the vector bundle pair (u^*TX, u^*TL) defined in the Appendix. In particular, if u is a (J, ν)-holomorphic map with closed genus g domain curve, then $\mathrm{ind}(D_{\mathbf{u}}) = \mathrm{ind}(\mathcal{D}_u) = 2c_1 + 2n(1-g)$, where c_1 is the first Chern number of the bundle $u^*TX \to \Sigma$.

Also we suppose Σ is biholomorphic to its conjugation $\bar{\Sigma}$, *i.e.* there exists an anti-holomorphic involution $c : \Sigma \to \Sigma$. Now for each $J \in \mathcal{J}_{\omega,\phi}$, we denote by $\mathcal{P}^J_{\phi,c}$ the set of $\nu \in \mathcal{P}$ satisfying $d\phi \circ \nu \circ dc = \nu$. Denote

$$\mathbb{J} := \{(J,\nu) | J \in \mathcal{J}_\omega, \nu \in \mathcal{P}\},$$

$$\mathbb{J}_\phi := \{(J,\nu) | J \in \mathcal{J}_{\omega,\phi}, \nu \in \mathcal{P}^J_{\phi,c}\} \subset \mathbb{J}.$$

We still assume L is relatively $Pin^\pm$ and fix a relative $Pin^\pm$ structure $\mathfrak{P}$ on L. If L is orientable, we fix an orientation on L.

Denote the orthogonal projection onto the normal bundle N_V by $\xi \mapsto \xi^N$. Since L_V, if nonempty, is a submanifold of L, we denote the normal bundle of L_V in L by N_{L_V}. Then for each (J,ν)-holomorphic open map u whose image lies in (V, L_V), the operator

$$D^N_u : \Gamma[(\Sigma, \partial\Sigma), (u^*N_V, u^*N_{L_V})] \to \Omega^{0,1}(\Sigma, u^*N_V),$$

$$D^N_{\mathbf{u}}(\xi) = [D_{\mathbf{u}}(\xi)]^N \tag{4.4}$$

by restricting the vertical linearization of $\bar{\partial}_{(J,\nu)}$ at u to the normal bundle, is also a real linear Fredholm operator.

Similar to the construction of relative GW invariants, we will restrict attention to a subspace $\mathbb{J}^V \subset \mathbb{J}$ ($\mathbb{J}^V_\phi \subset \mathbb{J}_\phi$ involving the involution) consisting of pairs (J,ν) that are compatible with V in the following sense.

Definition 4.1 *We say the pair $(J,\nu) \in \mathbb{J}$ is V-compatible if the following three conditions hold:*

(1) J preserves TV and $\nu^N|_V = 0$;

for all $\xi \in (N_V, N_{L_V})$, $v \in TV$ and $\vartheta \in T\Sigma$

(2) $[(\nabla_\xi J + J\nabla_{J\xi}J)(v)]^N = [(J\nabla_{Jv}J)\xi + (\nabla_v J)\xi]^N$;

(3) $[(J\nabla_{\nu(\vartheta)}J)\xi]^N = [(\nabla_\xi \nu + J\nabla_{J\xi}\nu)(\vartheta)]^N$.

We denote by $\mathbb{J}^V$ (resp. $\mathbb{J}^V_\phi$) the set of all V-compatible pairs $(J,\nu) \in \mathbb{J}$ (resp. $(J,\nu) \in \mathbb{J}_\phi$). The condition (1) in the Definition 4.1 ensures that V is a J-holomorphic submanifold, and that (J,ν)-holomorphic curves in $(V, L_V = V \cap L)$, if $V \cap L \neq \emptyset$, are also (J,ν)-holomorphic in (X,L). Conditions (2) and (3) ensure that for each (J,ν)-holomorphic map with closed domain curve whose image lies in V

$$u : \Sigma \to V,$$

the operator

$$\mathcal{D}^N_u : \Gamma[(\Sigma, u^*N_V) \to \Omega^{0,1}(\Sigma, u^*N_V),$$

$$\mathcal{D}^N_u(\xi) = [\mathcal{D}_u(\xi)]^N \tag{4.5}$$

by restricting the linearization of $\bar{\partial}_{(J,\nu)}$ at u to the normal bundle, is a complex linear operator. The following lemma is taken from [IP1].

Lemma 4.1 *Choose $(J,\nu) \in \mathbb{J}^V$ (resp. $\mathbb{J}^V_\phi$). Then for each (J,ν)-holomorphic map u from closed domain curve whose image lies in V, the operator $\mathcal{D}^N_u$ is a complex operator.*

Proof. It is enough to show that the operator commutes with J. We just need verify that for each $\xi \in N_V$, $[\mathcal{D}(J\xi) - J\mathcal{D}(\xi)]^N = 0$, since J preserves TV and so preserves the normal bundle. Note that

$$\mathcal{D}_u(\xi)(\vartheta) = \frac{1}{2}[\nabla_\vartheta \xi + J \circ \nabla_{j\vartheta}\xi + \nabla_\xi J \circ du \circ (j\vartheta)] - \nabla_\xi \nu(\vartheta).$$

Consider

$$\begin{aligned} 2J[\mathcal{D}(J\xi) - J\mathcal{D}(\xi)](\vartheta) &= (J\nabla_\vartheta J)\xi - (\nabla_{j\vartheta}J)\xi + (\nabla_\xi J + J\nabla_{J\xi}J) \circ du \circ (j\vartheta) \\ &\quad -2[\nabla_\xi \nu + J\nabla_{J\xi}\nu](\vartheta). \end{aligned}$$

Let $v = du \circ (j\vartheta) \in TV$. Note that $\nabla_\vartheta J = \nabla_{du\circ\vartheta} J$. Substituting the formula $du \circ \vartheta = 2\nu(\vartheta) - J \circ du \circ j\vartheta$ into the first term and taking the normal component, we see

$$\begin{aligned} 2J[\mathcal{D}(J\xi) - J\mathcal{D}(\xi)]^N(\vartheta) &= 2[(J\nabla_{\nu(\vartheta)}J)\xi - (\nabla_\xi \nu + J\nabla_{J\xi}\nu)(\vartheta)]^N \\ &\quad +[(\nabla_\xi J + J\nabla_{J\xi}J)(v)]^N - [(J\nabla_{Jv}J)\xi + (\nabla_v J)\xi]^N, \end{aligned}$$

then the conditions (2) and (3) in the Definition 4.1 ensure that the sum is zero. So $\mathcal{D}^N_u$ is a complex operator. $\square$

In the following we will not distinguish D and $\mathcal{D}$ and denote always by D the linearization operator. In the end of the section, we show a local normal form for open holomorphic maps near the points where they intersect V. Here the argument is similar to the Lemma 3.4 in [IP1].

Take a pair (J,ν) satisfying the condition (1) in the Definition 4.1. Let V be a codimension two J-holomorphic submanifold of X, and $u : (\Sigma, \partial\Sigma) \to (X, L)$ be a (J,ν)-holomorphic map which intersects V. Suppose $L_V = V \cap L \neq \emptyset$. Take a boundary marked point $p_{a\imath} \in (\partial\Sigma)_a$ and a interior marked point $q_\jmath$ satisfying $\mathbb{p} = u(p_{a\imath}) \in L_V$ and $\mathbb{q} = u(q_\jmath) \in V$. Let $\mathbb{H} = \{z = x + \mathrm{i}y |\ y \geq 0\}$ be the upper half complex plane. Fix a local holomorphic coordinate $z \in \mathbb{H}$ on a half open disc $\mathfrak{D}$ in Σ containing $p_{a\imath}$ or a local holomorphic coordinate on an open disc $\mathcal{O}$ in Σ containing $q_\jmath$. Also fix local coordinates $\{v^i\} = \{v^{i_1}, v^{i_2}\}$ ($1 \leq i_1 \leq n-1$, $n \leq i_2 \leq 2n-2$) in an open set $\mathscr{O}_V$ in V such that $\{v^{i_1}\}$ are the local coordinates in the open set $\mathscr{O}_V \cap L_V$ in L_V. And we extend $\{v^i\}$ to local coordinates $\{v^i, x\}$ for X with $x \equiv 0$ along V and so that $x = x^1 + \mathrm{i}x^2$ with $J(\frac{\partial}{\partial x^1}) = \frac{\partial}{\partial x^2}$ and $J(\frac{\partial}{\partial x^2}) = -\frac{\partial}{\partial x^1}$, moreover, we require that $\{v^{i_1}, x^1\}$ are just local coordinates for L.

Lemma 4.2 (normal form). *Suppose that Σ is a smooth connected Riemann surface with boundary and u is a (J,ν)-holomorphic map which intersects V at $\mathbb{p}$ or $\mathbb{q}$. Then either (1) $u(\Sigma) \subset V$, or (2) there exist an integer $K > 0$ and a nonzero $a_0 \in \mathbb{C}$ such that near interior intersection point*

$$u(z,\bar{z}) = (\mathbb{q}^i + O(|z|),\ a_0 z^K + O(|z|^{K+1})), \tag{4.6}$$

or near boundary intersection point

$$u(z,\bar{z}) = (\mathbb{p}^i + O(|z|),\ a_0 z^K + O(|z|^{K+1})), \tag{4.7}$$

in the local coordinates z and $\{v^i, x\}$, respectively, where $O(|z|^k)$ denotes a function of z and $\bar{z}$ that vanished to order k at $z = 0$. In particular, writing $z = r + \sqrt{-1}\ s$ and restricting (4.7) to the real part, we have

$$u(z, \bar{z})|_{\mathbb{R}} = u(r) = (\mathbb{p}^i + O(|r|),\ c_0 r^K + O(|r|^{K+1})), \tag{4.8}$$

in the local coordinates $\{v^{i_1}, x^1\}$ in L, where c_0 is a nonzero real number.

Proof. Let J_0 be the standard complex structure in the coordinates (v^i, x^α). Since J is close to J_0, we can set

$$A = (I - J_0 J)^{-1}(I + J_0 J) \quad \text{and} \quad \hat{\nu} = 2(I - J J_0)^{-1}\nu.$$

Then it is not hard to verify that the (J, ν)-holomorphic map equation $\bar{\partial}_J u = \nu$ is equivalent to

$$\bar{\partial}_{J_0} u = A \partial_{J_0} u + \hat{\nu}. \tag{4.9}$$

Note that the components of the matrix of J satisfy

$$(J - J_0)^i_j = O(|v| + |x|), \quad (J - J_0)^\alpha_\beta = O(|x|), \quad (J - J_0)^i_\alpha = O(|x|). \tag{4.10}$$

So we have

$$A^i_j = O(|v| + |x|), \quad A^\alpha_\beta = O(|x|), \quad A^i_\alpha = O(|x|), \quad \text{and} \quad \nu^\alpha = O(|x|)$$

since the normal component of ν vanishes along V. We express $u = (v^i(z, \bar{z}), x^\alpha(z, \bar{z}))$. Because A^i_α vanishes along V and dv^i and $\frac{\partial A^i_\alpha}{\partial x^\beta}$ are bounded near $p_{a\imath}$ or $q_\jmath$ we get

$$|dA^i_\alpha| \leq |\frac{\partial A^i_\alpha}{\partial v^j} \cdot dv^j + \frac{\partial A^i_\alpha}{\partial x^\beta} \cdot dx^\beta| \leq C(|x| + |dx|).$$

Also we have $|d\nu^\alpha| \leq C(|x| + |dx|)$. Now we analyse the equation (4.9), for the x components of u we have

$$\bar{\partial}_{J_0} x^\alpha = A^\alpha_i \partial_{J_0} v^i + A^\alpha_\beta \partial_{J_0} x^\beta + \hat{\nu}^\alpha, \tag{4.11}$$

thus

$$\partial_{J_0}\bar{\partial}_{J_0} x^\alpha = \partial_{J_0} A^\alpha_i \partial_{J_0} v^i + A^\alpha_i \partial^2_{J_0} v^i + \partial_{J_0} A^\alpha_\beta \partial_{J_0} x^\beta + A^\alpha_\beta \partial^2_{J_0} x^\beta + \partial_{J_0}\hat{\nu}^\alpha.$$

We have

$$|\Delta x^\alpha|^2 = |\frac{1}{2}\partial_{J_0}\bar{\partial}_{J_0} x^\alpha|^2 \leq C(|x|^2 + |\partial_{J_0} x|^2).$$

By Aronszajn's unique continuation theorem, if x^α vanishes to infinite order at $p_{a\imath}$ or $q_\jmath$ then $x^\alpha \equiv 0$ in a neighborhood of $p_{a\imath}$ or $q_\jmath$. That is to say, $u(\Sigma) \subset V$ locally. In fact, this conclusion is independent of coordinates, so we know $u(\Sigma) \subset V$. Otherwise, if the order K of vanishing is finite, then we can expand $x^\alpha(z, \bar{z})$ into Taylor series with beginning term $\sum_{k=0}^K \bar{z}^k z^{K-k}$. So x is $O(|z|^K)$, then from (4.11) we have

$$\bar{\partial}_{J_0} x^\alpha = O(|x|) = O(|z|^K).$$

But if the factor $\bar{z}$ appears in the order d item of Taylor expansion of $x^\alpha(z, \bar{z})$, $\bar{\partial}_{J_0} x^\alpha$ should be $O(|z|^{K-1})$. That is, we only have leading term $a_0 z^K$. So we get (4.6) and (4.7). □

5 Moduli space of V-regular open stable maps

Definition 5.1 *Given a codimension two symplectic submanifold V of (X, ω). A stable (J, ν)-holomorphic map $u : (\Sigma, \partial\Sigma) \to (X, L)$ is called V*-regular *if no component of its domain is mapped entirely into V and if neither any marked point nor any double point is mapped into V.*

The set of V-regular maps forms an open subset of the space of stable maps, denote it by $\mathcal{M}^V(X, L, \mathbf{d})$. Denote by $\mathcal{M}^V_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ the space of V-regular maps with marked points.

We denote by

$$\mathbb{r} = (r_{11}, \cdots, r_{1\mathbb{k}_1}, r_{21}, \cdots, r_{m1}, \cdots, r_{m\mathbb{k}_m}), \qquad \mathbb{s} = (s_1, \cdots, s_{\mathbb{l}})$$

the list of multiplicities of boundary and interior intersection points, respectively, where r_{ai} and s_j are positive integer numbers. And we define the *degree, length*, and *order* of $\mathbb{r}$ and $\mathbb{s}$ by

$$\deg \mathbb{r} := \sum_{a=1}^{m} \sum_{i=1}^{\mathbb{k}_a} r_{ai}, \quad \text{length}(\mathbb{r}) := \kappa = \sum_{a=1}^{m} \mathbb{k}_a, \quad \text{ord}(\mathbb{r}) := |\mathbb{r}| = \prod_{a=1}^{m} \prod_{i=1}^{\mathbb{k}_a} r_{ai};$$

$$\deg \mathbb{s} := \sum_{j=1}^{\mathbb{l}} s_j, \qquad \text{length}(\mathbb{s}) := \ell = \mathbb{l}, \qquad \text{ord}(\mathbb{s}) := |\mathbb{s}| = \prod_{j=1}^{\mathbb{l}} s_j.$$

The pair of vectors $(\mathbb{r}, \mathbb{s})$ labels the component of $\mathcal{M}^V_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$. We denote each $(\mathbb{r}, \mathbb{s})$-labeled component of $\mathcal{M}^V_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ by

$$\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d}) \subset \mathcal{M}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(X, L, \mathbf{d}),$$

where $\mathbf{k} + \mathbb{k}$ denotes the vector of numbers of boundary marked points $(k_1 + \mathbb{k}_1, \cdots, k_m + \mathbb{k}_m)$. Forgetting the additional $\kappa + \ell$ marked points defines a projection

$$\begin{array}{c} \mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d}) \\ \downarrow \\ \mathcal{M}^V_{\mathbf{k},l}(X, L, \mathbf{d}) \end{array} \tag{5.1}$$

onto one component of $\mathcal{M}^V_{\mathbf{k},l}(X, L, \mathbf{d})$, which is the disjoint union of such components. So for each fixed pair $(\mathbb{r}, \mathbb{s})$, (5.1) is a covering map to its image whose deck transformation group is the direct product of all groups of renumberings of the $\mathbb{k}_a$ marked points on each boundary $(\partial\Sigma)_a$ and the group of renumberings of the $\mathbb{l} = \ell$ interior marked points.

We define the universal moduli space

$$\mathcal{UM}_{\mathbf{k},l} = \mathcal{UM}_{\mathbf{k},l}(X, L, \mathbf{d}) := \{(\mathbf{u}, (J, \nu)) \mid (J, \nu) \in \mathbb{J}, \mathbf{u} \in \mathcal{M}_{\mathbf{k},l}(X, L, \mathbf{d})\}$$

consists of all (J, ν)-holomorphic maps representing $\mathbf{d}$, where (J, ν) varies over the space $\mathbb{J}$. The irreducible part of the universal moduli space involving the intersection data is denoted by $\mathcal{UM}^{*\,V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}$. Note that $\mathcal{UM}_{\mathbf{k},l}$ is also the zero set of the equation (4.1). We write the map by

$$\Phi(u, J, \nu) = \bar{\partial}_J u - \nu = \frac{1}{2}(du + J \circ du \circ j_\Sigma) - \nu. \tag{5.2}$$

Lemma 5.1 *For generic $(J,\nu)\in\mathbb{J}^V$, the irreducible part of $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\Bbbk,\mathbb{l}}(X,L,\mathbf{d})$ is a manifold with boundary of dimension*

$$\begin{aligned}\dim\ \mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\Bbbk,\mathbb{l}}(X,L,\mathbf{d}) &= \mu(d)+n(1-g)+(k+\kappa-\deg\ \mathbb{r})\\ &\quad +2(l+\ell-\deg\ \mathbb{s})-\dim Aut(\Sigma).\end{aligned} \tag{5.3}$$

Proof. It is a modification of the arguments used in the Lemma 4.2 and Lemma 4.3 of [IP1]. If we show that the irreducible part of universal moduli space $\mathcal{UM}^{*\ V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\Bbbk,\mathbb{l}}$ is a manifold, then the Sard-Smale transversality theorem and implicit function theorem imply that for generic $(J,\nu)\in\mathbb{J}^V$ the irreducible part of the V-regular moduli space is a manifold of the dimension (5.3).

Let $\mathfrak{M}^V_{\mathbf{k},l}$ denote the space of tuples $(u,J,\nu,\vec{z},\vec{w})$, where $(u,\vec{z},\vec{w})\in B^{1,p}_{\mathbf{k},l}(X,L,\mathbf{d})$, with u being taken to be V-regular and irreducible, and $(J,\nu)\in\mathbb{J}^V$. The map $\bar\partial_{J,\nu}$ is defined on $\mathfrak{M}^V_{\mathbf{k},l}$. We can apply the standard argument to show that the linearization $D\bar\partial_{J,\nu}$ is onto at the zeros set (see [RT2] for closed case). So the universal moduli space $\mathcal{UM}^{*\ V}_{\mathbf{k},l}=\bar\partial^{-1}_{J,\nu}(0)$ is smooth manifold with boundary, and similar to the Proposition 17.1 in [FOOO] for special case $g=0$ and $l=0$, and the Theorem C.10 in [McS], it is easy to calculate the dimension of $\mathcal{UM}^{*\ V}_{\mathbf{k},l}$ which is

$$\dim\mathcal{M}^V_{\mathbf{k},l}(X,L,\mathbf{d})=\mu(d)+n(1-g)+k+2l-\dim Aut(\Sigma).$$

Then we will show that the contact condition corresponding to each pair of ordered sequences $(\mathbb{r},\mathbb{s})$ is transverse. That will imply that the irreducible part $\mathcal{UM}^{*\ V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\Bbbk,\mathbb{l}}$ is a manifold.

Given nonnegative integers K,L. We denote by $\mathrm{Div}^{(K,L)}(\Sigma)$ the space associated with Σ which is the set of formal sums (with degree (K,L))

$$\sum_{a=1}^{m}\sum_i K_{ai}\delta_{ai}+\sum_j L_j\sigma_j,$$

where $\delta_{ai}\in(\partial\Sigma)_a$, $\sigma_j\in\Sigma$ and integers $K_{ai},L_j\geq 0$ with only finitely many of them being nonzero such that $\sum_{ai}K_{ai}=K$, $\sum_j L_j=L$. This is a smooth manifold of dimension $K+2L$.

For each pair of sequence $(\mathbb{r},\mathbb{s})$ such that deg $\mathbb{r}=K$, deg $\mathbb{s}=L$, let $\mathrm{Div}_{(\mathbb{r},\mathbb{s})}(\Sigma)\subset\mathrm{Div}^{(K,L)}(\Sigma)$ be the subset consisting of sums of the form

$$\sum_{a=1}^{m}\sum_{i=1}^{\Bbbk_a}r_{ai}x_{ai}+\sum_{j=1}^{\mathbb{l}}s_jy_j.$$

This is a smooth manifold of real dimension $\kappa+2\ell$.

Moreover, for each such pair $(\mathbb{r},\mathbb{s})$ define a map

$$\Upsilon_{(\mathbb{r},\mathbb{s})}:\mathcal{UM}^V_{\mathbf{k}+\Bbbk,l+\mathbb{l}}\longrightarrow\mathrm{Div}^{(K,L)}(\Sigma)\times\mathrm{Div}_{(\mathbb{r},\mathbb{s})}(\Sigma)$$

by

$$\Upsilon_{(\mathbb{r},\mathbb{s})}(u,J,\nu,\vec{z},\vec{w},\{x_{ai}\},\{y_j\})=(u^{-1}(V),\sum_{a=1}^{m}\sum_{i=1}^{\Bbbk_a}r_{ai}x_{ai}+\sum_{j=1}^{\mathbb{l}}s_jy_j)$$

where x_{ai}, y_j are the additional $\kappa+\ell$ intersection marked points. By Lemma 4.2, there are local holomorphic coordinates $z_{a\imath}$ on a half open disc $\mathfrak{D}$ in Σ containing $p_{a\imath}$ or $z_\jmath$ on an open disc $\mathcal{O}$ in Σ containing $q_\jmath$, and $u(p_{a\imath}) \in L_V \subset X$ and $u(q_\jmath) \in V \subset X$ such that the leading terms of the normal component of u are $z_{a\imath}^{K_{a\imath}}$ and $z_\jmath^{L_\jmath}$, respectively. That is

$$\Upsilon_{(\mathbb{r},\mathbb{s})}(u, J, \nu, \vec{z}, \vec{w}, \{x_{ai}\}, \{y_j\}) = (\sum_{a\imath} K_{a\imath} p_{a\imath} + \sum_{\jmath} L_\jmath\ q_\jmath\ , \sum_{ai} r_{ai} x_{ai} + \sum_j s_j y_j)$$

with $K_{a\imath}, L_\jmath \geq 1$, $\sum K_{a\imath} = K$, $\sum L_\jmath = L$. Let $\Delta \subset \mathrm{Div}^{(K,L)}(\Sigma) \times \mathrm{Div}_{(\mathbb{r},\mathbb{s})}(\Sigma)$ denote the diagonal of $\mathrm{Div}_{(\mathbb{r},\mathbb{s})}(\Sigma) \times \mathrm{Div}_{(\mathbb{r},\mathbb{s})}(\Sigma)$. Then

$$\mathcal{UM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}} = \Upsilon^{-1}_{(\mathbb{r},\mathbb{s})}(\Delta). \tag{5.4}$$

If $\Upsilon_{(\mathbb{r},\mathbb{s})}$ is transverse to Δ, then $\mathcal{UM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}$ is a manifold. Hence we only need to show that at each fixed $(u, J, \nu, \vec{z}, \vec{w}, \{x_{ai}\}, \{y_j\}) \in \mathcal{UM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ the differential $D\Upsilon$ is onto the tangent space of the first factor.

For this aim, similar to the method in [IP1], we construct a deformation

$$(u_t, J, \nu_t, \vec{z}, \vec{w}, \{x_{ai}\}, \{y_j\})$$

with

$$\bar{\partial}_J u_t = \nu_t. \tag{5.5}$$

We require that $(u_t, J, \nu_t, \vec{z}, \vec{w}, \{x_{ai}\}, \{y_j\})$ is tangent to $\mathcal{UM}^V_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}$ to first order in t, where the zeros of u_t^N are, to first order in t, the same as those of the polynomials $z_\iota^{K_\iota} + t\varphi_\iota(z_\iota)$ where φ_ι is an arbitrary polynomial in z_ι of degree less than K_ι defined near $z_\iota = 0$, and the degree K_ι equals to $K_{a\imath}$ or $L_\jmath$. For simplicity we consider fixed ι and omit it from the notation. In fact, it suffices to do this for $\varphi(z) = z^k$ for each $0 \leq k < K$ since the map $D\Upsilon$ is linear.

Choose smooth bump function β supported in small disc or half disc with $\beta \equiv 1$ in a neighborhood of $z = 0$. We also fix local coordinates $\{v^i\} = \{v^{i_1}, v^{i_2}\}$ for V around $u(0)$, and extend these to coordinates $\{v^i, x\}$ for X ($\{v^{i_1}, x^1\}$ for L) around $u(0)$ as described before in Lemma 4.2.

For any function $\eta(z)$ with $\eta(0) = 1$ (if around the boundary marked points, we further require that $\eta(r) \in \mathbb{R}$, *i.e.* the restriction of η to the real part is real), we construct maps

$$u_t = (u_0^T, u_0^N + t\beta z^k \eta). \tag{5.6}$$

The zeros of $u_0^N + t\beta z^k\eta$ have the form $z_t(1 + O(t))$, where z_t are the zeros of $z^K + tz^k$. Then the variation $\dot{u}$ at $t = 0$ is $\xi = \beta z^k \eta \mathfrak{N}_V$, where $\mathfrak{N}_V$ is a normal vector to V, in particular, if the omitted subscript $\iota = a\imath$, *i.e* $u(0)$ is the image of a boundary intersection marked point, then the normal vector $\mathfrak{N}_V$ is just the normal vector of L_V in L at $u(0)$.

Keeping J, z (or w), p (or q) fixed, we will show that we can suitably choose the function η and a variation $\dot{\nu}$ in ν such that $(\xi, 0, \dot{\nu}, 0, 0)$ is tangent to $\mathcal{UM}^V_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(X, L, \mathbf{d})$. That means the following two conditions are to be satisfied.

(1) If $(\xi, 0, \dot{\nu}, 0, 0)$ is tangent to the universal moduli space it must be in the kernel of the linearized operator of the map (5.2), thus, we have

$$D\xi(z) - \dot{\nu}(z, u(z)) = 0 \tag{5.7}$$

where D is given by

$$D_u(\xi) = \bar{\partial}_u \xi + \frac{1}{2} \nabla_\xi J \circ du \circ j - \nabla_\xi \nu.$$

(2) On the other hand, we must choose the variation $\dot{\nu}$ respecting to the V-compatible condition in Definition 4.1. That means $(0, \dot{\nu})$, which is the variation of (J, ν), must be tangent to $\mathbb{J}^V$. Thus $\dot{\nu}$ must satisfy the linearization of equations in Definition 4.1, *i.e.*

$$\dot{\nu}^N = 0, \qquad [(J\nabla_{\dot{\nu}(\cdot)} J)\mathfrak{N}_V]^N = [(\nabla_{\mathfrak{N}_V} \dot{\nu} + J\nabla_{J\mathfrak{N}_V} \dot{\nu})]^N(\cdot)$$

along V. These equalities hold if $\dot{\nu}$, in the local coordinates above, has an expansion near $x = 0$ of the form

$$\dot{\nu} = A(z, v) + B(z, v)\bar{x} + O(|z||x|) \tag{5.8}$$

satisfying $A^N = 0$ and $B^N = \frac{1}{2}[(J\nabla_{A(\partial/\partial z)} J)\mathfrak{N}_V]^N$.

To choose η and $\dot{\nu}$ satisfying (5.7) and (5.8), we first write $\dot{\nu} = \dot{\nu}^V + \dot{\nu}^N$, and take

$$\dot{\nu}^V = [D\xi]^V$$

along the graph $\{(z, v(z), z^K)\}$ of u_0 and extend it arbitrarily to a neighborhood of the origin. If we want to take $\dot{\nu}^N$ of the form (5.8), we have to solve the following equation locally around the origin

$$D^N \xi(z) = \dot{\nu}(z, v(z), z^K) = B^N([D\xi(0)]^V)\bar{z}^K + O(|z|^{K+1}) \tag{5.9}$$

where D^N is the operator in (4.4).

We can write $\xi = \alpha\mathfrak{N}_V + \gamma J\mathfrak{N}_V$ where α and γ are real.Identify it with $\xi = \zeta\mathfrak{N}_V$ where $\alpha{+}i\gamma$ is complex. When considering boundary marked point, recall that we take $\mathfrak{N}_V \in N_{L_V}$ and so $\gamma = 0$, $\xi = \alpha\mathfrak{N}_V$. The operator D is real linear, so we have

$$D\xi = (\bar{\partial}\zeta)\mathfrak{N}_V + \zeta E + \bar{\zeta} F \tag{5.10}$$

with

$$E = \frac{1}{2}[D(\mathfrak{N}_V) - JD(J\mathfrak{N}_V)], \qquad F = \frac{1}{2}[D(\mathfrak{N}_V) + JD(J\mathfrak{N}_V)].$$

Since we want to find a solution of the form $\zeta = \beta z^k \eta$ near the origin, we can take $\beta \equiv 1$. Then substituting $\zeta = z^k\eta$ into the equation (5.9) we get

$$- z^k \bar{\partial}\eta = z^k \eta E^N(z, \bar{z}) + \bar{z}^k \bar{\eta} F^N(z, \bar{z}) + B^N([D\xi(0)]^V)\bar{z}^K + O(|z|^{K+1}). \tag{5.11}$$

Note that when considering boundary marked point and restricting to the real part, the equation (5.11) restricts to real equation

$$- \frac{1}{2} r^k \frac{d\eta}{dr} = r^k \eta D^N(\mathfrak{N}_V) + B^N r^K + O(r^{K+1}). \tag{5.12}$$

When $k=0$, the equation (5.11) can always be solved by power series expansion. When $1 \le k < K$, we have $\zeta(0)=0$, so $B^N([D\xi(0)]^V)=0$ by (5.10). Then from the next Lemma, we know that F^N has the form of $az^{K-1}+O(|z|^K)$, so (5.11) is turned into

$$-\bar{\partial}\eta = \eta E^N(z,\bar{z}) + a\bar{z}^k\bar{\eta}z^{K-1-k} + O(|z|^{K+1-k}),$$

which can also be solved by power series. Moreover, when restricting to the real part, the solution would obey the equation (5.12), that implies that the η is desired. □

Lemma 5.2 *Near the origin, $F^N = az^{K-1}+O(|z|^K)$ for some constant a.*

Proof. Fix a vector γ tangent to the domain of u. Denote $\Gamma = du(\gamma)$. Then using the formula (4.3) and the (J,ν)-holomorphic map equation $du(\gamma) = 2\nu(\gamma) - J\circ du \circ j(\gamma)$, we calculate

$$\begin{aligned}
4F^N(\mathfrak{N}_V)(\gamma) &= 4F^N(\Gamma,\gamma) \\
&= J(\nabla_\Gamma J)\mathfrak{N}_V - (\nabla_{J\Gamma}J)\mathfrak{N}_V + (\nabla_{\mathfrak{N}_V}J)J\Gamma \\
&\quad + J(\nabla_{J\mathfrak{N}_V}J)J\Gamma + 2(\nabla_{J\nu(\gamma)})\mathfrak{N}_V - 2(\nabla_{\mathfrak{N}_V}J)J\nu(\gamma) \\
&\quad - 2(\nabla_{J\mathfrak{N}_V}J)\nu(\gamma) - 2(\nabla_{\mathfrak{N}_V}\nu)\gamma - 2J(\nabla_{J\mathfrak{N}_V}\nu)\gamma. \qquad (5.13)
\end{aligned}$$

Note that the normal component of $\Gamma = du(\gamma)$ is $Kz^{K-1}\frac{\partial}{\partial x}+O(|z|^K)$. Thus we only need to calculate (5.13) for the components of Γ in the V direction, since the difference is of the form $z^{K-1}\Phi_1(z,\bar{z})$. Note also that

$$J(v(z),z^K) = J(v(z),0) + O(|z|^K),$$

and

$$\nabla J = (\nabla J)(v(z),0) + O(|z|^K).$$

So for Γ tangent to V and for J, ∇J taking values at $(v(z),0)$, by the V-compatibility conditions in Definition 4.1, we can verify that (5.13) vanishes. This completes the proof.□

6 Limits of V-regular open maps

We come to construct a compactification of each component of the moduli space of V-regular open maps. Using this compactification and the method by Solomon of defining open GW invariants, we can define relative open invariant for symplectic-Lagrangian pair (X,L), where the Lagrangian submanifold L is the fixed point set of an anti-symplectic involution ϕ, of dimension 3 if L is orientable and of dimension 2 if L might not be orientable (under some additional assumptions).

To compactify $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d})$, we take its closure

$$\mathcal{CM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d}) \qquad (6.1)$$

in the stable moduli space $\overline{\mathcal{M}}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(X,L,\mathbf{d})$. In fact, the closure lies in the subset of $\overline{\mathcal{M}}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(X,L,\mathbf{d})$ consisting of open stable maps whose last $\kappa+\ell$ marked points are

mapped into V, still with associated multiplicities $(\mathbb{r},\mathbb{s})$, although the actual order of contact might be infinite.

We will show that the closure is an orbifold with boundary. That is to prove that the frontier $\mathcal{CM}^V \setminus \mathcal{M}^V$ is a subset of codimension at least 1. Since such frontier is a subset of the space of stable maps, it is stratified according to the type of bubble structure of the domain. The following proposition is the main result in this section describing the structure of the closure $\mathcal{CM}^V$.

Proposition 6.1 *For generic pair $(J,\nu) \in \mathbb{J}^V$, each stratum of the irreducible part of*

$$\mathcal{CM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d}) \setminus \mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d})$$

is a manifold of dimension at least one less than the dimension (5.3) of $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d})$.

We will prove this proposition for different cases, according to the different types of limits:

Case 1. stable maps with no components or special ((**k**,l)-marked and double) points lying entirely in V;

Case 2. if $L \cap V \neq \emptyset$, a stable open map with smooth domain which is mapped entirely into V;

Case 3. stable maps with some components in V and some off V.

Case 1. The analysis is essentially similar to the arguments in [FOOO] and [Liu]. Since we do not admit the degeneration of domain, each stratum of this type is labeled by the pair of positive integer numbers $(\mathbb{b},\mathbb{i})$ of boundary and interior double points of their nodal domain curve $\widehat{\Sigma}$. For such fixed $\widehat{\Sigma}$, the corresponding stratum is denoted by $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d})_{\widehat{\Sigma}}$.

Lemma 6.1 *In the 'Case 1', for generic pair $(J,\nu) \in \mathbb{J}^V$, the irreducible part of the stratum $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d})_{\widehat{\Sigma}}$ of $\mathcal{CM}^V$ is a manifold of dimension $\mathbb{b}+2\mathbb{i}$ less than the dimension (5.3) of $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d})$.*

Proof. Let $\widetilde{\Sigma} \to \widehat{\Sigma}$ be the normalization of $\widehat{\Sigma}$. Then $\widetilde{\Sigma}$ is a smooth curve with a pair of marked points corresponding to each double point of $\widehat{\Sigma}$. We will prove that the irreducible part of $\mathcal{UM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d})_{\widehat{\Sigma}}$, denoted simply by $\mathcal{UM}^*_{\widehat{\Sigma}}$, is a codimension $\mathbb{b}+2\mathbb{i}$ submanifold of the irreducible part of $\mathcal{UM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d})$, denoted simply by $\mathcal{UM}^*_{\widetilde{\Sigma}}$. Then the Sard-Smale theorem implies the conclusion.

For simplicity, we consider the special case that there is only one pair of such boundary marked points (z_1,z_2). We can define evaluation map at these two points

$$evb : \mathcal{UM}^*_{\widetilde{\Sigma}} \to L \times L,$$

then $\mathcal{UM}^*_{\widehat{\Sigma}}$ is the inverse image of the diagonal Δ in $L \times L$. Since by the arguments in the proof of the Lemma 5.1 $\mathcal{UM}^*_{\widetilde{\Sigma}}$ is a manifold, we only need to verify that the evaluation map is transverse to Δ.

For this purpose, we fix an arbitrary $(\mathbf{u}_0,(J,\nu)) \in evb^{-1}(\Delta)$ and choose local coordinates in X around $p = u_0(z_1) = u_0(z_2)$, and choose bump functions β_1 and β_2 supported

in small half discs around z_1 and z_2. Then we deformation u_0 locally around z_1 by $u_t = u_0 + t\beta_1 \mathbb{v}$ and locally around z_2 by $u_t = u_0 - t\beta_2 \mathbb{v}$, where $\mathbb{v}(z) \in TX$ around p such that $\mathbb{v}|_{\mathbb{R}} \in T_pL$. Also modify ν so that $\nu_t = \bar{\partial} u_t$. Then the derivative of this path u_t at $t = 0$ is a tangent vector $\mathfrak{x}$ to $\mathcal{UM}^*_{\Sigma}$ with $evb_*(\mathfrak{x}) = (\mathbb{v}(0), -\mathbb{v}(0))$. So evb is transverse to Δ. The dimension argument is direct since the degree of freedom of $z_i \in \partial\Sigma$ is one. $\square$

Case 2. If $L_V = L \cap V \neq \emptyset$, denote by $\mathcal{SM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ the subset of strata of $\mathcal{CM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ consisting of equivalence classes of open maps with smooth domain whose image is contained in (V, L_V). We see that

$$\mathcal{SM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d}) = \mathcal{CM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d}) \cap \mathcal{M}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(V, L_V, \mathbf{d}). \tag{6.2}$$

We will use the linearized operator to characterize the maps in $\mathcal{SM}^V$.

For each $u \in \mathcal{M}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(V, L_V, \mathbf{d})$, denote by D^V the linearization of the equation $\bar{\partial} u = \nu$ at the map u. Note that there is a surjective restriction map

$$\mathbb{J}^V \to \mathbb{J}(V)$$

which maps a V-compatible pair (J, ν) on X to its restriction to V. Then by Lemma 5.1, for generic $(J, \nu) \in \mathbb{J}^V$, the irreducible part of $\mathcal{M}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(V, L_V, \mathbf{d})$ is a smooth manifold of dimension

$$\text{index } D^V = \mu_V(d) + (n-1)(1-g) + (k+\kappa) + 2(l+\ell) - \text{dim}Aut(\Sigma). \tag{6.3}$$

We consider several operators related to the maps in $\mathcal{M}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(V, L_V, \mathbf{d})$. First, we have the linearization $D^X_{(\mathbb{r},\mathbb{s})}$ of the equation $\bar{\partial} u = \nu$. The domain space of $D^X_{(\mathbb{r},\mathbb{s})}$ consists of sections of (u^*TX, u^*TL) that satisfy the linearization of V-contact conditions specified by the pair of sequences $(\mathbb{r}, \mathbb{s})$, and with index given by (5.3).

Secondly, there is an operator D^N similar to (4.4) obtained by applying D^X to vector fields normal to V and then projecting back onto the subspace of normal vector fields. It is a bounded operator on the Sobolev completion $W^{p,2}(u^*N_V, u^*N_{L_V})$ with p derivatives in L^2, *i.e.*

$$D^N : \ W^{p,2}((\Sigma, \partial\Sigma), (u^*N_V, u^*N_{L_V})) \to W^{p-1,2}(T^*\Sigma \otimes u^*N_V). \tag{6.4}$$

For $p > \deg \mathbb{r} + \deg \mathbb{s}$, we have a closed subspace $W^{p,2}_{(\mathbb{r},\mathbb{s})}(u^*N_V, u^*N_{L_V})$ consisting of sections satisfying the linearization of V-contact conditions specified by $(\mathbb{r}, \mathbb{s})$. Then denote by $D^N_{(\mathbb{r},\mathbb{s})}$ the restriction of D^N to that subspace $W^{p,2}_{(\mathbb{r},\mathbb{s})}$. We can calculate the index of $D^N_{(\mathbb{r},\mathbb{s})}$ by

$$\begin{aligned} \text{index } D^N_{(\mathbb{r},\mathbb{s})} &= \text{index } D^X_{(\mathbb{r},\mathbb{s})} - \text{index } D^V \\ &= \mu_{N_V}(d) + 1 - g - \deg \mathbb{r} - 2\deg \mathbb{s} \\ &= 1 - g \end{aligned} \tag{6.5}$$

since $\deg \mathbb{r} + 2\deg \mathbb{s} = \mu_{N_V}(d)$.

The following lemma is a key observation

Lemma 6.2 *Each element of $\mathcal{SM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ is a map into V with* $\ker D^N_{(\mathbb{r},\mathbb{s})} \neq 0$.

Proof. We adopt the renormalization argument similar to ones in [IP1] and [T]. Suppose there exists a sequence $\{u_n\}$ in $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d})$ converges to $u\in\mathcal{M}_{\mathbf{k},+\mathbb{k},l+\mathbb{l}}(V,L_V,\mathbf{d})$. Since in this case no bubbling occurs, $u_n\to u$ in C^∞. When n is large enough, the images of maps u_n lie in a neighborhood of V, so we can identify them with a subset in the normal bundle N_V of V by the exponential map. Denote by $\mathbb{u}_n$ the projection of u_n to V along the fibers of N_V, so that $\mathbb{u}_n\to u$ in C^∞.

Then denote by $R_t: N_V\to N_V$ the dilation with a factor of $\frac{1}{t}$, $t\in\mathbb{R}^+$. For each n, there exist a unique $t=t_n$ such that the C^1-norm of the normal component of the map

$$R_n(u_n):=R_{t_n}(u_n)$$

is 1. Obviously, $t_n\to 0$. Let $U_n=R_n(u_n)$ denote these renormalization maps. They are holomorphic with respect to the renormalized pair $(R_n^*J,R_n^*\nu)$, *i.e.*

$$\bar\partial_{R_n^*J}U_n-R_n^*\nu=R_n^*(\bar\partial_J u_n-\nu)=0. \tag{6.6}$$

We can see that

$$(R_n^*J,R_n^*\nu)\to(J_0,\nu_0)$$

in C^∞, where the limit (J_0,ν_0) is dilation invariant and equals to the restriction of (J,ν) along V. Also the sequence $\{U_n\}$ is bounded in C^1. So we can apply the classic elliptic bootstrapping techniques and obtain a converged subsequence $U_n\to U_0$ satisfying

$$\bar\partial_{J_0}U_0=\nu_0.$$

The map U_n can be written as $\exp_{\mathbb{u}_n}\xi_n$ where $\xi_n\in\Gamma(\mathbb{u}_n^*N_V,\mathbb{u}_n^*N_{L_V})$ is identified with the normal component of U_n, whose C^1-norm is 1. Then $U_n\to U_0$ implies that $\xi_n\to\xi\in\Gamma(u^*N_V,u^*N_{L_V})$ in C^∞, and ξ is a nonzero section since the normal component of U_0 has C^1-norm equal to 1. Note that the limit ξ has zeros specified by $(\mathbb{r},\mathbb{s})$ since $\{u_n\}$ are under the contact constraints described by $(\mathbb{r},\mathbb{s})$. Thus we only need to show $D_u^N\xi=0$.

Denote by T_n the parallel transport along the curves $\exp_{\mathbb{u}_n}(t\xi_n)$, $0\le t\le t_n$. And we can write $u_n=\exp_{\mathbb{u}_n}(t_n\xi_n)$. For each fixed n, u_n and $\mathbb{u}_n$ have the same domain, then by the definition of linearization

$$T_n^{-1}(\bar\partial_J u_n-\nu_{u_n})-(\bar\partial_J\mathbb{u}_n-\nu_{\mathbb{u}_n})=D_{\mathbb{u}_n}(t_n\xi_n)+O(|t_n\xi_n|^2).$$

Since u_n is (J,ν)-holomorphic, the first term in the above equation vanishes. And the Definition 4.1(1) implies that the normal component of $\bar\partial_J\mathbb{u}_n-\nu_{\mathbb{u}_n}$ vanishes since the image of the projection $\mathbb{u}_n$ lies in V. Dividing the equation above by t_n and taking limits of normal components, we obtain

$$D_u^N\xi=\lim_{n\to\infty}D^N_{\mathbb{u}_n}(\xi_n)=0,$$

since $|\xi_n|_{C^1}=1$. That means $\ker D^N_{(\mathbb{r},\mathbb{s})}\neq 0$. □

The operator $D^N_{(\mathbb{r},\mathbb{s})}$ depends only on the 1-jet of $(J,\nu)\in\mathbb{J}^V$, thus we can consider the restriction map

$$\mathbb{J}^V\to\mathbb{J}^1 \tag{6.7}$$

which maps a V-compatible pair (J, ν) on X to its 1-jet along V.

Denote by $\mathcal{UM}(V, L_V, \mathbf{d})$ the universal moduli space of all open pseudo-holomorphic maps into (V, L_V). This space is a fiber bundle over $\mathbb{J}^1$. Its elements are denoted by $(\mathbf{u}, J, \nu)$. Then we can define a bundle

$$\begin{array}{c} \text{Fred} \\ \downarrow \\ \mathcal{UM}(V, L_V, \mathbf{d}) \end{array} \tag{6.8}$$

where Fred is the bundle whose fiber at u is the space of all real linear Fredholm maps (6.4) of index $\iota \leq 0$. We can consider $D^N_{(\mathbb{r},\mathbb{s})}$ as a smooth section of (6.8) if index $D^N_{(\mathbb{r},\mathbb{s})} \leq 0$. Since the arguments by Koschorke [K] can apply to the real Fredholm operators, we see that

$$\text{Fred} = \bigcup_k \text{Fred}_k$$

where Fred_k is the codimension $k(k-\iota)$ submanifold consisting of all the operators with k dimensional kernel. Note that, at an operator D, the normal bundle to Fred_k in Fred is $\text{Hom}(\ker D, \text{coker} D)$.

Lemma 6.3 *The section $D^N_{(\mathbb{r},\mathbb{s})}$ of (6.8) is transverse to each* Fred_k.

Proof. Fix $(u, J, \nu) \in \mathcal{UM}(V, L_V, \mathbf{d})$ such that the operator $D^N_{(\mathbb{r},\mathbb{s})}$ at (u, J, ν) lies on Fred_k. To verify the transversality, it suffices to show that all elements in $\text{Hom}(\ker D^N_{(\mathbb{r},\mathbb{s})}, \text{coker} D^N_{(\mathbb{r},\mathbb{s})})$ can be realized by some variations. In fact, for any nonzero elements $\mathbb{k} \in \ker D^N_{(\mathbb{r},\mathbb{s})}$ and $\mathbb{c} \in \ker(D^N_{(\mathbb{r},\mathbb{s})})^*$, we aim to find a variation in ν such that

$$\int_\Sigma \langle \mathbb{c}, (\delta D^N_{(\mathbb{r},\mathbb{s})})\mathbb{k} \rangle \ \text{dvol} \neq 0,$$

since, by Theorem C.1.10 in [McS], $(\delta D^N_{(\mathbb{r},\mathbb{s})})\mathbb{k} \in \text{coker} D^N_{(\mathbb{r},\mathbb{s})}$. In the inequality above, the brackets mean the inner product on the domain Σ and $(D^N)^*$ denotes the L^2 formal adjoint operator

$$(D^N)^* : \ W^{p,2}(T^*\Sigma \otimes u^*N_V) \to W^{p-1,2}((\Sigma, \partial\Sigma), (u^*N_V, u^*N_{L_V})).$$

Denote by π^N the projection onto the normal bundle. Note that $D^N_{(\mathbb{r},\mathbb{s})} = \pi^N \circ D^X_{(\mathbb{r},\mathbb{s})}$, so

$$(\delta D^N_{(\mathbb{r},\mathbb{s})})\mathbb{k} = (\delta\pi^N) D^X_{(\mathbb{r},\mathbb{s})}\mathbb{k} + \pi^N(\delta D^X)\mathbb{k} + \pi^N D^X(\delta\pi^N)\mathbb{k},$$

where the operator D^X is given by (4.3). We want to take the variation with (u, J) fixed and ν varying as $\nu_t = \nu + t\mu$ with $\mu \equiv 0$ along V. Thus π^N is fixed and we get

$$\int_\Sigma \langle \mathbb{c}, (\delta D^N_{(\mathbb{r},\mathbb{s})})\mathbb{k} \rangle \ \text{dvol} = \int_\Sigma \langle \mathbb{c}, (\delta D^X)\mathbb{k} \rangle \ \text{dvol} = -\int_\Sigma \langle \mathbb{c}, \nabla_{\mathbb{k}}\mu \rangle \ \text{dvol}.$$

Take a point $x \in \Sigma$ such that $\mathbb{k}(x) \neq 0$. Choose a neighborhood W of x in Σ and a neighborhood U of $u(x)$ in X such that the section $\mathbb{k}$ restricting to W has no zeros in

U. Extending $\mathbb{c}$ to a smooth section $\tilde{\mathbb{c}}$ of $\mathrm{Hom}(T\Sigma, TX)$ along $W \times U$ such that $\tilde{\mathbb{c}}|_V$ is a section of $\mathrm{Hom}(T\Sigma, N_V)$.

Then we construct the $(0,1)$ form μ such that its 1-jet along V satisfies

$$\mu|_V = 0,\ \nabla_{\mathbb{K}(y)}\mu(x,y) = (\beta\tilde{\mathbb{c}})(x,y)$$

and

$$\nabla_{J\mathbb{K}(y)}\mu(x,y) = -J(\beta\tilde{\mathbb{c}})(x,y),$$

where β is a smooth bump function supported on $W \times U$ with $\beta \equiv 1$ on a slightly small open set. Since μ vanishes along V, the left-hand side of the formula (3) in Definition 4.1 vanishes, so the compatibility conditions are satisfied. Furthermore,

$$\int_\Sigma \langle \mathbb{c}, (\delta D^N_{(\mathbb{r},\mathbb{s})})\mathbb{K}\rangle \ \mathrm{dvol} = -\int_\Sigma \langle \mathbb{c}, \nabla_{\mathbb{K}}\mu\rangle \ \mathrm{dvol} = -\int_U \beta|\mathbb{c}|^2 \ \mathrm{dvol}.$$

Note that $\mathbb{c} \in \ker(D^N_{(\mathbb{r},\mathbb{s})})^*$, *i.e.* it is the solution of an elliptic equation, so the unique continuation theorem for elliptic operators implies that $|\mathbb{c}|$ can not vanish on any open set. Then the lemma is proved. $\square$

Lemma 6.4 *In the 'Case 2', $\mathcal{SM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d})$ is contained in the space*

$$\mathcal{M}'(V,L_V,\mathbf{d}) = \{\mathbf{u} \in \mathcal{M}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(V,L_V,\mathbf{d}) \mid \ker D^N_{(\mathbb{r},\mathbb{s})} \neq 0\}. \tag{6.9}$$

Furthermore, for generic $(J,\nu) \in \mathbb{J}^V$, the irreducible part of $\mathcal{M}'(V,L_V,\mathbf{d})$ is a submanifold of the irreducible part of $\mathcal{M}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(V,L_V,\mathbf{d})$ of at least dimension 1 less than (5.3).

Proof. Obviously, Lemma 6.2 implies the first conclusion. Then, note that the equality

$$\mathrm{index}\ D^N_{(\mathbb{r},\mathbb{s})} = \dim\ \mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d}) - \dim\ \mathcal{M}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(V,L_V,\mathbf{d}),$$

if $\mathrm{index}\ D^N_{(\mathbb{r},\mathbb{s})} > 0$, by (6.5) $\mathrm{index}\ D^N_{(\mathbb{r},\mathbb{s})} = 1$, the second statement is trivial. So we just assume that $\iota = \mathrm{index}\ D^N_{(\mathbb{r},\mathbb{s})} \leq 0$.

Denote by

$$\mathcal{UM}' = (D^N_{(\mathbb{r},\mathbb{s})})^{-1}(\mathrm{Fred} \setminus \mathrm{Fred}_0)$$

the set of elements $(\mathbf{u},J,\nu) \in \mathcal{UM}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(V,L_V,\mathbf{d})$ for which $D^N_{(\mathbb{r},\mathbb{s})}$ has a nontrivial kernel. Then by Lemma 6.3, $\mathcal{UM}'$ is a codimension $(1-\iota)$ subset of $\mathcal{UM}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(V,L_V,\mathbf{d})$, and in fact a submanifold off a set of codimension $2(2-\iota)$. Since the projection

$$\pi : \mathcal{U}M' \to \mathbb{J}^1$$

is Fredholm, the Sard-Smale theorem implies that for a second category set of $(J,\nu) \in \mathbb{J}^1$ the fiber (6.9)

$$\pi^{-1}(J,\nu) = \mathcal{M}'(V,L_V,\mathbf{d})$$

is a manifold of dimension

$$\begin{aligned} \mathrm{index}\ D^V - (1-\iota) &= \mathrm{index}\ D^V + \mathrm{index}\ D^N_{(\mathbb{r},\mathbb{s})} - 1 \\ &= \mathrm{index}\ D^X_{(\mathbb{r},\mathbb{s})} - 1. \end{aligned}$$

The inverse image of this second category set under (6.7) is also a second category set in $\mathbb{J}^V$. That is to say, for generic $(J,\nu) \in \mathbb{J}^V$, the irreducible part of (6.9) is a submanifold of codimension at least one in $\mathcal{CM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X,L,\mathbf{d})$. $\square$

Case 3. Now we consider the last case that there is a sequence of V-regular open maps converges to a limit map $u \in \mathcal{CM}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ whose domain is the union $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 are bubble domains of genus g_1 and g_2, and u restricts to a V-regular map $u_1 : \Sigma_1 \to X$ and a holomorphic map $u_2 : \Sigma_2 \to V$ into V. Limit maps of this type arise from sequences of V-regular maps in which either (1) two of the $(\mathbb{k}, \mathbb{l})$-contact intersection points collide in the domain or (2) one of the original $(\mathbf{k}, l)$-marked points whose image sinks into V, collides with a contact point. In either case the collison produces a ghost bubble map $u_2 : \Sigma_2 \to V$ whose energy is at least $\hbar_V$ by Lemma 2.1.

Then note that $u_1^{-1}(V)$ consists of the nodal points $\Sigma_1 \cap \Sigma_2$ and some of the last $\kappa + \ell$ marked points $p_{a\imath} \in \partial\Sigma$, $q_\jmath \in \Sigma$. The nodes are defined by identifying points $x_j \in \Sigma_1$ and $y_j \in \Sigma_2$. Since u_1 is V-regular and $u_1(x_j) \in V$ (resp. in L_V if $L_V \neq \emptyset$), then Lemma 4.2 associates a multiplicity s'_j (resp. r'_j) to each x_j, the multiplicity vector is denoted by

$$\mathbb{s}' = (s'_1, s'_2, \cdots) \qquad (\text{resp.} \quad \mathbb{r}' = (r'_1, r'_2, \cdots)\ ).$$

For convenience, images of these nodes x_j and y_j is denoted by vectors $(\mathbb{k}', \mathbb{l}')$. Similarly, since u arises as a limit of V-regular maps, the $q_\jmath$ (resp. $p_{a\imath}$), which are limits of the contact points with V (resp. L_V), also have associated multiplicities. We split the set of $q_\jmath$ (resp. $p_{a\imath}$) into the points $\{q^1_\jmath\}$ (resp. $\{p^1_{a\imath}\}$) on Σ_1 (resp. $\partial\Sigma_1$) and $\{q^2_\jmath\}$ (resp. $\{p^2_{a\imath}\}$) on Σ_2 (resp. $\partial\Sigma_2$), denote by

$$\mathbb{s}^1 = (s^1_1, s^1_2, \cdots) \qquad \text{and} \qquad \mathbb{s}^2 = (s^2_1, s^2_2, \cdots)$$

$$(\text{resp.} \quad \mathbb{r}^1 = (\cdots, r^1_{a\imath}, \cdots) \qquad \text{and} \qquad \mathbb{r}^2 = (\cdots, r^2_{a\imath}, \cdots)\ )$$

the associated multiplicity vectors. Hence we write u as a pair

$$u = (u_1, u_2) \in \mathcal{M}^{V,\mathbb{r}^1+\mathbb{r}',\mathbb{s}^1+\mathbb{s}'}_{g_1,\mathbf{k}_1,l_1,\mathbb{k}_1+\mathbb{k}',\mathbb{l}_1+\mathbb{l}'}(X, L, \mathbf{d}_1) \times \overline{\mathcal{M}}_{g_2,\mathbf{k}_2+\mathbb{k}_2+\mathbb{k}',l+\mathbb{l}_2+\mathbb{l}'}(V, L_V, \mathbf{d}_2) \tag{6.10}$$

where $\mathbf{d}_i = [u_i]$ with $\mathbf{d}_1 + \mathbf{d}_2 = \mathbf{d}$ satisfying the matching conditions $u_1(x_j) = u_2(y_j)$, and $(\mathbf{k}_1, l_1) + (\mathbf{k}_2, l_2) = (\mathbf{k}, l)$.

Lemma 6.5 *In this case 3, the only elements (6.10) lying in $\mathcal{CM}^V_{(\mathbb{r},\mathbb{s})}$ are those for which there exists a (singular) section $\xi \in \Gamma((\Sigma, \partial\Sigma), (u_2^* N_V, u_2^* N_{L_V}))$ nontrivial on at least one component of Σ_2 with zeros of order $r^2_{a\imath}$ at $p^2_{a\imath} \in (\partial\Sigma)_a$, $a = 1, \cdots, m$, or of order $s^2_\jmath$ at $q^2_\jmath \in \Sigma$, poles of order r'_j or s'_j at y_j, and $D^N_{u_2}\xi = 0$ where $D^N_{u_2}\xi$ is as in (4.4).*

We still use a renormalization argument similar to the one used in Lemma 6.2. Since we consider poles of the section, we will discuss the renormalization in a compactification $\mathbb{P}_V$ of the normal bundle $\pi : N_V \to V$. Before starting the proof, we recall the description of $\mathbb{P}_V$ given in [IP1].

First, note that N_V is a complex line bundle with an inner product and a compatible connection induced by the Riemannian connection on X. In fact, we have

$$\pi_{\mathbb{P}} : \mathbb{P}_V = \mathbb{P}(N_V \oplus \mathbb{C}) \to V, \qquad \mathbb{P}_{L_V} = \mathbb{P}(N_{L_V} \oplus \mathbb{R}) \to L_V,$$

i.e. $\mathbb{P}_V$ is the fiberwise complex projection of the Whitney sum of N_V with the trivial complex line bundle.

We define the bundle map

$$\iota : N_V \hookrightarrow \mathbb{P}_V$$

fiberwise by

$$\iota(x) = [x, 1].$$

It is a embedding onto the complement of the infinity section $V_\infty \subset \mathbb{P}_V$. The multiplication map $R_t(\xi) = \xi/t$ on N_V defines a $\mathbb{C}^*$ action on $\mathbb{P}_V$.

When V is a point we can identify $\mathbb{P}_V$ with $\mathbb{P}^1$ and give it the Kähler structure $(\omega_\varepsilon, g_\varepsilon, j)$ of the 2-sphere of radius ε. Then $\iota : \mathbb{C} \hookrightarrow \mathbb{P}_V$ is a holomorphic map with

$$\iota^* g_\varepsilon = \phi_\varepsilon^2 [(dr)^2 + r^2 (d\theta)^2]$$

and

$$\iota^* \omega_\varepsilon = \phi_\varepsilon^2 r dr \wedge d\theta = d\psi_\varepsilon \wedge d\theta$$

where

$$\phi_\varepsilon(r) = \frac{2\varepsilon}{1+r^2} \qquad \text{and} \qquad \psi_\varepsilon(r) = \frac{2\varepsilon^2 r^2}{1+r^2}.$$

This construction globalizes by interpreting r as the norm on the fibers of N_V, replacing $d\theta$ by the connection 1-form α on N_V and including the curvature F_α of that connection. So we have a closed form

$$\iota^* \omega_\varepsilon = \pi^* \omega_V + \psi_\varepsilon \pi^* F_\alpha + d\psi_\varepsilon \wedge \alpha$$

which is nondegenerate for small ε and whose restriction to each fiber of N_V agrees with the volume form on the 2-sphere of radius ε.

At each point $p \in N_V$ the connection determines a horizontal subspace which identifies $T_p N_V$ with the fiber of $N_V \oplus TV$ at $\pi(p)$. The fibers of N_V have a complex structure j_0 and a metric g_0, and restrict J and g on X to V, then one can verify that for small ε the forms

$$\omega_\varepsilon, \quad \tilde{J} = j_0 \oplus J|_V, \quad \text{and} \quad \tilde{g}_\varepsilon = (\phi_\varepsilon^2 g_0) \oplus g|_V$$

extend over V_∞ to define a tamed triple $(\omega_\varepsilon, \tilde{J}, \tilde{g}_\varepsilon)$ on $\mathbb{P}_V$. Note that Lemma 2.1 also holds for tamed structures, and so we can choose sufficiently small ε such that every (J, ν)-holomorphic map u from $\mathcal{S} = S^2$ or D^2 onto a fiber of $\mathbb{P}_V \to V$ of degree $d \leq [u_1] \cdot V$ satisfies

$$\int_{\mathcal{S}} |du|^2 \leq \frac{\hbar_V}{8} \tag{6.11}$$

where $\hbar$ is the constant associated with V by Lemma 2.1. We fix such an ε and write ω_ε as $\omega_{\mathbb{P}}$. Denote by V_0 the zero section of $\mathbb{P}_V$.

Then we identify an $\varepsilon' < \varepsilon$ neighborhood of V_0 in $\mathbb{P}_V$ with a neighborhood of $V \subset X$ and pullback (J, g) from X to $\mathbb{P}_V$. Fix a bump function β supported on the ε' neighborhood of V_0 with $\beta \equiv 1$ on the $\varepsilon'/2$ neighborhood. For each small $t > 0$ let $\beta_t = \beta \circ R_t$. Starting with the metric $h_t = \beta_t g + (1 - \beta_t) g_\varepsilon$, using the procedure described in the appendix of [IP1], we can obtain a compatible $(\omega_{\mathbb{P}}, J_t, g_t)$ on $\mathbb{P}_V$. Then as $t \to 0$ we have $J_t \to \tilde{J}$ in C^0 on $\mathbb{P}_V$ and $g_t \to g_0$ on compact sets of $\mathbb{P}_V \setminus V_\infty$.

Proof of Lemma 6.5. Suppose that a sequence of V-regular open maps $u_n : \Sigma_n \to X$ converges to $u = (u_1, u_2)$ as above. That is, the domains Σ_n converge to $\Sigma = \Sigma_1 \cup \Sigma_2$, by Theorem 2.1, u_n converge to $u : \Sigma \to X$ in C^0 and C^∞ away from the double points of Σ, preserving the energy. For the interior double points, the argument is the same as the one in [IP1], so for simplicity we only consider the boundary double points.

Around each double point $x_j = y_j$ of $\partial\Sigma_1 \cap \partial\Sigma_2$ we have coordinates $(z_j, w_j) \in \mathbb{H}^2$ in which Σ_n is locally the locus of $\{z_j w_j = \mu_{j,n}\}$ and Σ_1 is locally $\{z_j = 0\}$. Let $A_{j,n}$ be the half-annuli in the neck of the bordered neck of Σ_n defined by

$$A_{j,n} = \{|\mu_{j,n}|/\delta \le |z_j| \le \delta\}.$$

Denote by $\Sigma'_n \subset \Sigma_n$ the bordered neck $A_n = \cup_j A_{j,n}$ together with everything on the Σ_2 side of A_n, and denote the restriction of u_n to Σ'_n by u'_n.

The restriction of u_n to $\Sigma'_n \setminus A_n$ converge to u_2. By Lemma 2.1, there exists a constant $\hbar_V$ depending only on V such that the modified energy $\mathbb{E}(u_2) > \hbar_V$, since the image of u_2 lies in V. Then we fix δ small enough so that the modified energy of $u = (u_1, u_2)$ inside the union of δ-balls around the double points is at most $\hbar_V/32$. for sufficiently large n

$$\mathbb{E}(\pi_V \circ u_n|_{\Sigma'_n \setminus A_n}) \ge \hbar_V/2 \tag{6.12}$$

and

$$\mathbb{E}(u_n|_{A_n}) \le \hbar_V/16.$$

Note that for large n the image of u'_n lies in a tubular neighborhood of V which is identified with a neighborhood of V_0 in $\mathbb{P}_V$. Using renormalization $R_t : N_V \to N_V$ and the bundle map $\iota : N_V \hookrightarrow \mathbb{P}_V$, u'_n gives rise to a one-parameter family of maps $\iota \circ R_t \circ u'_n$ into $\mathbb{P}_V$. We can consider the modified energy (2.6) of the corresponding map

$$\iota \circ R_t \circ u'_n : \Sigma'_n \to \mathbb{P}_V$$

on the part of the domain which is mapped into the upper hemisphere $\mathbb{P}^+_V$ calculated using the metric $\tilde{g}_\varepsilon$ on $\mathbb{P}_V$ constructed above. The energy is close to zero for large t and exceeds $\hbar_V/4$ for small t by (6.12). Therefore there exists a unique $t = t_n$ such that the maps

$$g_n = \iota \circ R_{t_n} \circ u_n : \Sigma'_n \to \mathbb{P}_V$$

satisfy

$$\mathbb{E}(g_n|_{g_n^{-1}(\mathbb{P}^+_V) \cup A_n}) = \frac{\hbar_V}{4}. \tag{6.13}$$

Noting (6.12), $t_n \to 0$ as $n \to \infty$ since $u_n(\Sigma'_n \setminus A_n) \to V$ pointwise.

Next, consider the small bordered annuli $B_{j,n}$ in the bordered neck of Σ_n defined by $\{\delta/2 \le |w_j| \le \delta\}$ and let $B_n = \cup_j B_{j,n}$. On each $B_{j,n}$, u_n converges in C^1 to

$$u_1 = (a_j w_j^{r'_j} + \cdots)$$

and $u_n(B_{j,n})$ is of small diameter. Thus, after passing to the subsequence and taking smaller δ if necessary, each $g_n(B_{j,n})$ lies in a coordinate neighborhood V_j centered at a

point $q_j \in V_\infty$ with $\mathrm{diam}^2(V_j) < \hbar_V/1000$. Fix a smooth bump function β on Σ_n which is supported on Σ'_n satisfying $0 \le \beta \le 1$ and $\beta \equiv 1$ on $\Sigma'_n \setminus B_n$, and such that on each $B_{j,n}$

$$\int_{B_{j,m}} |d\beta|^2 \le 100. \tag{6.14}$$

Then we extend Σ'_n to a new curve with boundary in the Lagrangian submanifold, that is to smoothly attach a half disc H_j along the half circle $\gamma_{j,n} = \{|w_j| = \delta\}$. Extend g_n to

$$\bar{g}_n : \overline{\Sigma} = \Sigma'_n \cup \{H_j\} \to \mathbb{P}_V$$

by setting $\bar{g}(H_j) = q_j$ and coning off g_n on $B_{j,n}$ by $\bar{g}_n = \beta \cdot g_n$ in the coordinates on V_j. The local expansion of u_1 shows that $u_n(\gamma_{j,n})$, oriented by the coordinate w_j, has winding number r'_j around V_0. The same is true for $g_n(\gamma_{j,n})$. So in relative homology, $[\bar{g}_n]$ is $\iota_*[u_2] + r'\mathcal{F}$ where $r' = \sum r'_j$ and $\mathcal{F}$ is the fiber class of $(\mathbb{P}_V, \mathbb{P}_{L_V}) \to (V, L_V)$.

Then by (6.13) and (6.14), the energy of $\bar{g}_n$ on the region that is mapped into $\mathbb{P}_V^+$ is bounded by

$$\int_{g_n^{-1}(\mathbb{P}_V^+)} |dg_n|^2 + \sum_j \mathrm{diam}^2(V_j) \int_{B_{j,n}} |d\beta|^2 \le \frac{\hbar_V}{2}. \tag{6.15}$$

On the other hand, in the region mapped into $\mathbb{P}_V^-$, $\bar{g}_n = g_n$ is (J_n, ν_n)-holomorphic with $J_n \to \tilde{J}$ and $\nu_n \to \pi^*\nu_V$, so the energy in that region is dominated by its symplectic are (2.3). So we have

$$\begin{aligned}
\mathbb{E}(\bar{g}_n|_{g_n^{-1}(\mathbb{P}_V^-)}) &\le \frac{\hbar_V}{2} + c_1 \int_{g_n^{-1}(\mathbb{P}_V^-)} g_n^*\omega_{\mathbb{P}} \\
&\le \frac{\hbar_V}{2} + c_1\omega_{\mathbb{P}} \cdot [\bar{g}_n(\overline{\Sigma}_n)] + c_1 \Big| \int_{g_n^{-1}(\mathbb{P}_V^+)} \bar{g}_n^*\omega_{\mathbb{P}} \Big|.
\end{aligned} \tag{6.16}$$

Combining with (6.15) we have a uniform energy bound

$$\mathbb{E}(\bar{g}_n) \le c_1\omega \cdot ([u_2] + r'F) + c_2.$$

Also the restrictions g'_n of g_n to $\Sigma'_n \setminus B_n$ have such energy bound. These g'_n are (J_n, ν_n)-holomorphic maps, so we apply Theorem 2.1 to obtain a subsequence which converges to a $(\tilde{J}, \pi^*\nu_V)$-holomorphic map whose domain is Σ_2 together with the half-discs $\{|w_j| \le \delta/2\}$ in Σ_1 and some possible bubble components.

After deleting these half-discs, the limit is a map

$$g_0 : \widetilde{C}_2 \to \mathbb{P}_V$$

with $g_0(y_j) \in V_\infty$ at marked points y_i. Note that the projection $\pi \circ g'_n$ converge to u_2, so the irreducible components are divided into two types: (1) Those components biholomorphically identified with components of C_2 on which g_0 is a lift of u_2 to $\mathbb{P}_V$ and, (2) those mapped by g_0 into fibers of $\mathbb{P}_V$ which are unstable bubbles and can be collapsed by the stabilization. Then (6.15) implies that no type (1) component is mapped into V_∞. Since the type (2) components are $(J, 0)$-holomorphic, then by (6.12) these components contribute to the integral (6.15) at most $\hbar_V/8$. So at least one component of type (1) is not mapped into V_0 by g_0.

Lemma 4.6 says that each component of g_0 has a local expansion normal to V_∞ given by $b_j z_j^{d_j} + \cdots$ at each $y_j \in \partial\Sigma_2$. Note that $\partial A_{j,n} \setminus L = \gamma_{j,n} \cup \gamma'_{j,n}$, where $\gamma'_{j,n}$ is the half-circle $\{|z_j| = \delta\}$ oriented by z_j. Thus d_j is the local winding number of $g_n(\gamma'_{j,n})$ with V_∞, which is equal to the local winding number of $g_n(\gamma_{j,n})$ with V_∞, that is just r'_j.

Since $g'_n \to g_0$ on Σ_2, the sections ξ_n of $u_2^* N_V$, satisfying $\exp(\xi_n) = \iota^{-1} g_n$, converge to nonzero ξ satisfying $\exp(\xi) = \iota^{-1} g_0$. Then by the same arguments in the proof of Lemma 6.2, $D^N_{u_2}\xi = 0$, and the calculation above implies that ξ has a pole or order r'_j at each boundary double points y_j. Moreover, g_n have the same zeros and multiplicities as u_n, so the zeros of ξ are exactly the last $\kappa + \ell$ marked points of the limit curve Σ_2 with the original multiplicity pair of vectors $(\mathbb{r}, \mathbb{s})$. $\square$

7 The space of V-stable open maps

From last section we see that the limit of a sequence of V-regular open maps is a stable map whose components are of the types described in Case 1-3. Actually, the components of the limit map are also partially ordered according to the rate at which they sink into V. In this section we will make this precise by introducing a *layer structure* on the domain, and then construct a compactification of the space of V-regular maps.

Let Σ be a stable curve with boundary.

Definition 7.1 *A layer structure on Σ is the assignment of a non-negative integer λ_j to each irreducible component Σ_j to Σ, such that at least one component must have $\lambda_j = 0$ or 1.*

The union of all the components with $\lambda_j = K$ is called the *layer K stable curve*, denoted by Λ_K. Note that Λ_K might not be a connected curve.

Definition 7.2 *A marked layer structure on a $(\mathbf{k} + \mathbb{k}, l + \mathbb{l})$-marked bordered stable curve $\Sigma_{\mathbf{k},l,\mathbb{k},\mathbb{l}}$ is a layer structure on Σ together with*

(1) *a pair of vectors $(\mathbb{r}, \mathbb{s})$ recording the multiplicities of the last $\kappa + \ell$ boundary and interior intersection marked points, and*

(2) *a pair of vectors $(\mathbb{a}, \mathbb{b})$ which assigns multiplicities to each boundary double point of $\partial\Lambda_K \cap \partial\Lambda_L$, and to each interior double point of $\Lambda_K \cap \Lambda_L$, $K \neq L$.*

Note that double points within a layer are not assigned a multiplicity.

Each layer Λ_K then has boundary points $p_{K,\imath}$ of type (1) with multiplicity vector $\mathbb{r}_K = (r_{K,\imath})$ (here we do not distinguish the different boundary components) and interior points $q_{K,\jmath}$ of the same type with multiplicity vector $\mathbb{s}_K = (s_{K,\jmath})$, and has double pints with multiplicities. The double points in Λ_K that are assigned multiplicities are divided into two types. Let $\mathbb{a}_K^+$ (resp. $\mathbb{b}_K^+$) be the vector derived from $\mathbb{a}$ (resp. $\mathbb{b}$) that gives the multiplicities of the boundary (resp. interior) double points $y_{K,i}^+$ where Λ_K meets the higher layers at boundary (resp. interior part), *i.e.* the points $\partial\Lambda_K \cap \partial\Sigma_j$ (resp. $\Lambda_K \cap \Sigma_j$) with $\lambda_j > K$. Let $\mathbb{a}_K^-$ (resp. $\mathbb{b}_K^-$) be the similar vector of multiplicities of boundary (resp. interior) double points $y_{K,i}^-$ where Λ_K meets the lower layers.

Then we associate operators D_K^N similar to (6.4) defined on the layers Λ_K, $K \geq 1$, as follows. For the interior double points, the argument is the same as the one in [IP1], so

we only consider the special case that we assume there are only boundary double points. For each choice of $\mathbb{a} = \mathbb{a}_K^- = \{\alpha_{K,i}^-\}$ and ρ, fix smooth weighting function $W_{\mathbb{a},\rho}$ which has the form $|z_i|^{\rho+\alpha_{K,i}^-}$ in some local coordinates z_i in a small half-disc centered at $y_{K,i}^-$ and has no other zeros. Then given a stable map $u : \Lambda_K \to V$, denote by $L^m_{\mathbb{a}_K^-,\delta}(u^*N_V, u^*N_{L_V})$ the Hilbert space of all L^m_{loc} sections of $(u^*N_V, u^*N_{L_V})$ over $(\Lambda_K, \partial\Lambda_K) \setminus \{y_{K,i}^-\}$ which are finite in the norm

$$\|\xi\|^2_{m,\mathbb{a},\delta} = \sum_{l=1}^{m} \int_{\Lambda_K} |W_{\mathbb{a},l+\delta} \cdot \nabla^l \xi|^2.$$

For large m the elements ξ in this space have poles with $|\xi| \leq c|z_i|^{-\alpha_{K,i}^- - \delta}$ at each $y_{K,i}^-$ and have $m-1$ continuous derivatives elsewhere on Λ_K. For such m, denote by $L^m_{K,\delta}(u^*N_V, u^*N_{L_V})$ the closed subspace of $L^m_{\mathbb{a}_K^-,\delta}(u^*N_V, u^*N_{L_V})$ consisting of all sections that vanish to order $r_{K,j}$ and $s_{K,j}$ at $p_{K,j}$ and $q_{K,j}$ and order $\alpha_{K,i}^+$ at $y_{K,i}^+$. Hence by elliptic theory for weighted norms, the operator D^N defines a bounded operator

$$D_K^N : L^m_{K,\delta}((\Lambda_K, \partial\Lambda_K), (u^*N_V, u^*N_{L_V})) \to L^{m-1}_{K,\delta+1}(T^*\Lambda_K \otimes u^*N_V). \tag{7.1}$$

For generic $0 < \delta < 1$, D_K^N is Fredholm with

$$\begin{aligned} \text{index } D_K^N &= \mu_{N_V}([u(\Lambda_K)]) + \chi(\Lambda_K) + (\deg \mathbb{a}_K^- - \deg \mathbb{r}_K - \deg \mathbb{a}_K^+) \\ &\quad +2(\deg \mathbb{\beta}_K^- - \deg \mathbb{s}_K - \deg \mathbb{\beta}_K^+) \\ &= \chi(\Lambda_K) \end{aligned} \tag{7.2}$$

where $\chi(\Lambda_K)$is the Euler characteristic of Λ_K, and the index formula would be more general for cases admitting interior double points. The formula is derived from the fact that the Euler class of the pair of complex line bundle and totally real sub-bundle $(u^*N_V, u^*N_{L_V})$ can be computed from the zeros and poles of a section.

Then we give the following definition of the stable maps we want.

Definition 7.3 *A V-stable map is a stable map $u \in \overline{\mathcal{M}}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(X, L, \mathbf{d})$ together with*

(1) *a marked layer structure on its domain Σ with $u|_{\Lambda_0}$ being V-regular, and*

(2) *for each $K \geq 1$, an element $\xi_K \in \ker D_K^N$ defined on the layer Λ_K that is a section nontrivial on every irreducible component of Λ_K.*

Denote by $\overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ the $(\mathbb{r}, \mathbb{s})$-labeled component of the set of all V-stable open maps. This contains the set $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ of V-regular maps as the open subset, *i.e.* the V-stable maps whose entire domain lies in layer 0. Forgetting the data ξ_K defines a map

$$\mathscr{F} : \overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d}) \to \overline{\mathcal{M}}_{\mathbf{k}+\mathbb{k},l+\mathbb{l}}(X, L, \mathbf{d}). \tag{7.3}$$

Note that the number of layers of a V-stable map must be finite. Assume there are altogether $r+1$ layers $\Lambda_0, \Lambda_1, \cdots, \Lambda_r$. Each V-stable map $(\mathbf{u}, \xi_1, \cdots, \xi_r)$ determines

an element of the space $\mathcal{H}^V_{(X,L)}$ [4] as follows. For sufficiently small ε, we can push the components in V off V by composing u with $\exp(\varepsilon^K \xi_K)$ and for each K, smoothing the domain at the nodes $\Lambda_K \cap (\cup_{L>K}\Lambda_L)$ and smoothly joining the images where the zeros of $\varepsilon^K\xi_K$ on Λ_K approximate the poles of $\varepsilon^{K+1}\xi_{K+1}$. The resulting map

$$u_\xi = u|_{\Lambda_0} \# \exp(\varepsilon\xi_1) \# \cdots \# \exp(\varepsilon^r \xi_r)$$

is V-regular and represents a homology class $\mathscr{H}(\mathbf{u}, \xi_K) = \mathscr{H}(u_\xi) \in \mathcal{H}^V_{(X,L)}$. Note that this class is independent of the choice of the small ε, and depends on each ξ_K up to a nonzero multiplier.

Thus, we have a well-defined map

$$\mathscr{H} : \overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d}) \to \mathcal{H}^{V,\mathbb{r},\mathbb{s}}_{(X,L),\mathbf{d}}. \tag{7.4}$$

The following proposition is an open version similar to the Proposition 7.3 in [IP1].

Proposition 7.1 *There exists a topology on $\overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ such that this space of V-stable maps is compact and the maps $\mathscr{F}$ of (7.3) and $\mathscr{H}$ of (7.4) are continuous and differential on each stratum.*

Proof. We analyze a sequence of V-regular maps and a more general sequence of V-stable maps to define the topology on $\overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$.

Let $\{u_n : \Sigma_n \to X\}$ be a sequence of maps in $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$. By the bubble convergence Theorem 2.1, there is a subsequence, still denoted by u_n, which converges to a stable map $u : \Sigma \to X$. We claim that the argument of renormalization in the last section will define the structure of a V-stable map on the limit map.

First, note that the last $\kappa + \ell$ marked points converge, so the pair of intersection multiplicity vectors $(\mathbb{r}, \mathbb{s})$ of u_n is invariant in the limit, that is the pair of vectors in Definition 7.2(1). Next, we inductively define the rest of the layer structure. Let $\lambda_j = 0$ on each component Σ_j which is not mapped into V by u, thus we have the 0 layer Λ_0. Denote by $\Sigma(1)$ the union of those remaining components which are mapped into V. We assign to each boundary (resp. interior) double point $y \in \Lambda_0 \cap \Sigma(1)$ a multiplicity α_y (resp. β_y) which is equal to the order of contact of $u|_{\Lambda_0}$ with V at y.

Then Lemma 6.5 gives the sections $\xi_{n,1}$ (and the renormalized maps $g_{n,1} = \exp \xi_{n,1}$) which converge to a nontrivial element of $\xi_1 \in \ker D^N$ on $\Sigma(1)$. We assign $\lambda_j = 1$ to each component in $\Sigma(1)$ on which ξ_1 is nonzero, and denote the union of the remaining components by $\Sigma(2)$. Thus the Λ_1 is defined and we obtain the section ξ_1 that is nonzero on each component of Λ_1. Furthermore,

(1) ξ_1 vanishes at the boundary (resp. interior) double points y where $\Lambda_1 \cap \Sigma(2)$. We assign such y a multiplicity α_y (resp. β_y) equal to the order of vanishing of ξ_1 at y.

(2) The proof of Lemma 6.5 shows that ξ_1 has a pole at $x \in \Lambda_0 \cap \Lambda_1$ of order α_x (or β_x), and has zeros of order r_{1i} (or s_{1i}) at the intersection points in Λ_1.

[4]Here the definition of $\mathcal{H}^V_{(X,L)}$ is similar to the one of $\mathcal{H}^V_X$ appeared in the Definition 5.1 of [IP1]. Moreover, similar to the map (5.9) of [IP1], we may get a well-defined map $\mathscr{H} : \mathcal{M}^V_{\mathbf{k},l}(X, L) \to \mathcal{H}^V_{(X,L)}$, which records the homology-intersection data.

(3) Since $\xi_{n,1} \to \xi_1$, the argument above shows that, for large n, u_n and

$$u|_{\Lambda_0} \# \exp(\varepsilon \xi_1) \# g_{n,1}|_{\Sigma(2)}$$

gives the same homology class in $\mathcal{H}^V_{(X,L)}$.

So we define the section ξ_1 and the associated two pairs of vectors $(\mathbb{r}, \mathbb{s})$ and $(\mathbb{\alpha}, \mathbb{\beta})$ on Λ_1.

Then we repeat the procedure of construction on $\Sigma(2)$ and continue from the case on $\Sigma(K)$ to the case on $\Sigma(K+1)$. That defines inductively a layer structure on Σ, pairs of multiplicity vectors $(\mathbb{\alpha}, \mathbb{\beta})$ for each point in $\Lambda_K \cap \Sigma(K+1)$, and a nontrivial section $\xi_K \in \ker D^N_K$ on each layer Λ_K. We only need to repeat finite r times because each $\Sigma(K)$ has fewer components than $\Sigma(K-1)$ and the total number of components of a stable map is finite. Finally, we have a nontrivial ξ_K on each component of Λ_K, $K \geq 1$. Moreover, the limit V-stable map $(u, \xi_1, \cdots, \xi_r)$ defines an element in $\mathcal{H}^{V,\mathbb{r},\mathbb{s}}_{(X,L),\mathbf{d}}$.

Then we come to consider sequences in $\overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$. Given a sequence $U_n = (u_n, \xi_n)$ of V-stable maps, denote each layer by Λ^n_K, we first define $u_{n,0} = u_n|_{\Lambda^n_0}$. Since the sequence $\{u_{n,0}\}$ has a uniform energy bound, applying the discussion above produces a subsequence converging to a V-stable map U^0. Similarly, the $u_{n,1} = u_n|_{\Lambda^n_1}$ converge to a stable map u_1 into V and the renormalized maps $\exp(\xi_{n,1}) : \Lambda^n_1 \to \mathbb{P}_V$ have a subsequence converging to a limit $\exp(\xi_{0,1})$ with $\xi_{0,1} \in \ker D^N_1$. Then $U^1 = (u_1, \xi_{0,1})$ is a V-stable map with image lying in V and its bottom layer coincides with the top layer of U^0 to form a V-stable map $U^0 \cup U^1$. This process continues finite (say m) times since each layer carries energy at least $\hbar_V$. Hence, there exists a subsequence of $\{U_n\}$ which converges to a V-stable map $U^0 \cup \cdots \cup U^m$.

Actually, the above procedure defines a notion of convergence of a sequence of V-stable maps. Such convergence defines a topology on $\overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$. So in this sense, we see that $\overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ is compact and the map $\mathscr{H}$ is continuous with this topology. $\square$

The next theorem is the key result needed to define the relative invariants, which implies the Proposition 6.1.

Theorem 7.1 *The space of V-stable maps is compact and there exists a continuous map*

$$\varepsilon_V : \overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d}) \xrightarrow{\mathbf{ev} \times \mathscr{H}} L^{|\mathbf{k}|} \times X^l \times \mathcal{H}^{V,\mathbb{r},\mathbb{s}}_{(X,L),\mathbf{d}}. \tag{7.5}$$

Moreover, the complement of $\mathcal{M}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ in the irreducible part of $\overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \mathbf{d})$ has codimension at least one.

Proof. The discussion above implies compactness. For computing the dimension, we recall the operator (7.1). We can use the bubble domain curves $\Sigma = \cup \Lambda_K$ to label the strata of the space of V-stable maps, and denote these strata by $\overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(\Sigma)$. For each layer Λ_K, $K \geq 1$, denote by

$$\overline{\mathcal{M}}^{V,\mathbb{r}_K \cup \mathbb{\alpha}^+_K \cup \mathbb{\alpha}^-_K, \mathbb{s}_K \cup \mathbb{\beta}^+_K \cup \mathbb{\beta}^-_K}_{\Lambda_K, \mathbf{k}_K, l_K}(X, L, [\mathbf{u}(\Lambda_K)]) \tag{7.6}$$

the set of V-stable maps (u, ξ) such that the restriction

$$u|_{\Lambda_K} \in \mathcal{M}_{\Lambda_K, k_K + \mathcal{L}(\mathbb{r}_K \cup \mathbb{\alpha}^+_K \cup \mathbb{\alpha}^-_K), l_K + \mathcal{L}(\mathbb{s}_K \cup \mathbb{\beta}^+_K \cup \mathbb{\beta}^-_K)}(V, L_V, [\mathbf{u}(\Lambda_K)])$$

is a map from Λ_K to V so that the operator (7.1) has a nontrivial kernel on each irreducible component of Λ_K, where the notation $\mathcal{L}(\cdot)$ denotes the length of an intersection multiplicity vector.

Under such setting we consider the Hilbert bundle over the universal moduli space

$$L^m_K(V) \to \overline{\mathcal{UM}}_{\Lambda_K, k_K+\mathcal{L}(\mathbb{r}_K\cup\alpha^+_K\cup\alpha^-_K), l_K+\mathcal{L}(\mathbb{s}_K\cup\beta^+_K\cup\beta^-_K)}(V, L_V, [\mathbf{u}(\Lambda_K)])$$

whose fiber at $u:\Lambda_K\to V$ is the space $L^m_{K,\delta}((\Lambda_K, \partial\Lambda_K), (u^*N_V, u^*N_{L_V}))$ in (7.1). Then we can use the same argument as the proof of Lemma 6.3 to show that D^N_K defines a section of

$$\begin{array}{c} \mathrm{Fred}(L^m_{K,\delta}(V), L^{m-1}_{K,\delta}(V)) \\ \Big\downarrow \\ \overline{\mathcal{UM}}_{\Lambda_K, k_K+\mathcal{L}(\mathbb{r}_K\cup\alpha^+_K\cup\alpha^-_K), l_K+\mathcal{L}(\mathbb{s}_K\cup\beta^+_K\cup\beta^-_K)}(V, L_V, [\mathbf{u}(\Lambda_K)]) \end{array} \tag{7.7}$$

which is transverse to every codimension $k(k-\iota)$ subspace Fred_k of Fredholm operators with kernel of dimension $k\geq 1$. Then the following two lemmas gives the dimension statements of the theorem. $\Box$

Lemma 7.1 *The irreducible part of the space (7.6) is a manifold of dimension at most*

$$\begin{aligned} \mathrm{d}^X_K &= \mu_X([u(\Lambda_K)]) + (n-3)\chi(\Lambda_K) + k_K + 2l_K \\ &\quad + \mathcal{L}(\alpha^-_K) + \mathcal{L}(\mathbb{r}_K) + \mathcal{L}(\alpha^+_K) + (\deg\,\alpha^-_K - \deg\,\mathbb{r}_K - \deg\,\alpha^+_K) \\ &\quad + 2[\mathcal{L}(\beta^-_K) + \mathcal{L}(\mathbb{s}_K) + \mathcal{L}(\beta^+_K) + \deg\,\beta^-_K - \deg\,\mathbb{s}_K - \deg\,\beta^+_K] - 1. \end{aligned} \tag{7.8}$$

Proof. By Theorem 2.2 (2), generically

$$\mathcal{M}_{\Lambda_K, k_K+\mathcal{L}(\mathbb{r}_K\cup\alpha^+_K\cup\alpha^-_K), l_K+\mathcal{L}(\mathbb{s}_K\cup\beta^+_K\cup\beta^-_K)}(V, L_V, [\mathbf{u}(\Lambda_K)])^*$$

is a manifold of dimension

$$\begin{aligned} \mathrm{d}^V_K &= \mu_V([u(\Lambda_K)]) + (n-3)\chi(\Lambda_K) + k_K + 2l_K \\ &\quad + \mathcal{L}(\alpha^-_K) + \mathcal{L}(\mathbb{r}_K) + \mathcal{L}(\alpha^+_K) + 2[\mathcal{L}(\beta^-_K) + \mathcal{L}(\mathbb{s}_K) + \mathcal{L}(\beta^+_K)]. \end{aligned} \tag{7.9}$$

Comparing (7.2),(7.8) and (7.9), we see that $\mathrm{d}^X_K - \mathrm{d}^V_K = \iota - 1$ where $\iota = \mathrm{index} D^N_K$. If $\iota > 0$ the lemma holds.

If $\iota \leq 0$, we can still use the theorem of Koschorke and conclude that the set of $u \in \mathcal{M}_{\Lambda_K}(V, L_V, [\mathbf{u}(\Lambda_K)])^*$ with $\dim\ker D^N_K = k$ form a submanifold of codimension $k(k-\iota) \geq (1-\iota) \geq 1$ since $k \geq 1$. Since by Definition 7.3 (2) the section ξ_K is nontrivial on every component of Λ_K, the lemma holds. $\Box$

Remark. If $\partial\Lambda_K = \emptyset$, then d^X_K is just like the computing result in [IP1] which is

$$\begin{aligned} \mathrm{d}^X_K &= 2c_1([u(\Lambda_K)]) + (n-3)\chi(\Lambda_K) + 2l_K \\ &\quad + 2[\mathcal{L}(\beta^-_K) + \mathcal{L}(\mathbb{s}_K) + \mathcal{L}(\beta^+_K) + \deg\,\beta^-_K - \deg\,\mathbb{s}_K - \deg\,\beta^+_K] - 2. \end{aligned}$$

where c_1 denotes the first Chern class of TX.

Lemma 7.2 *Each irreducible stratum* $\overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(\Sigma)$ *is a manifold of dimension* $r^b + \mathcal{L}^b_K + 2(r^i + \mathcal{L}^i_K)$ *less than (5.3), where* r^b *(resp.* r^i*) is the total number of nontrivial layers with nonempty boundary (resp. without boundary) and* $\mathcal{L}^b_K$ *(resp.* $\mathcal{L}^i_K$*) is the total number of boundary (resp. interior) double points, whether or not intersecting other layers, in layer* Λ_K*,* $K \geq 0$*.*

Proof. Note that the stratum $\overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(\Sigma)$ is the product of the spaces (7.6) for each layer constrained by the matching conditions $u(x) = u(y)$ at each of the $\mathcal{L}(\alpha) + \mathcal{L}(\beta)$ double points where Λ_K meets other layers. Then we can use the transversality argument in [McS] to show that the irreducible part of this space is a manifold of expected dimension, the dimension has a upper bound

$$\dim \overline{\mathcal{M}}^{V,\mathbb{r},\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(\Sigma) \leq \sum_{K=0}^{r} \mathrm{d}^X_K - \mathcal{L}(\alpha) - 2\mathcal{L}(\beta). \tag{7.10}$$

We also notice the following facts :

(1) the formula (7.8) is additive in the homology class $u(\Lambda_K)$ and the multiplicity vectors $\mathbb{r}_K$ and $\mathbb{s}_K$. And we have the equalities

(2) $\sum_{K=0}^r \deg \alpha^+_K = \sum_{K=0}^r \deg \alpha^-_K$, and $\sum_{K=0}^r \deg \beta^+_K = \sum_{K=0}^r \deg \beta^-_K$,

(3)

$$\sum_{K=0}^{r} \chi(\Lambda_K) = \chi(\Sigma) + \mathcal{L}(\alpha) + 2\mathcal{L}(\beta)$$

where $\mathcal{L}(\alpha) = \sum \mathcal{L}(\alpha^+_K) = \sum \mathcal{L}(\alpha^-_K)$, $\mathcal{L}(\beta) = \sum \mathcal{L}(\beta^+_K) = \sum \mathcal{L}(\beta^-_K)$.
Then substituting the (7.8) in (7.10), the dimension statements is verified. □

Remark. As the reason mentioned before, in the present paper, we will not use the map (7.5) to define invariants which involve more delicate homology-intersection data. Instead, we only care about the simplest intersection data, so in the following, to define simpler invariants, we only use the evaluation maps at both marked and intersection points.

8 Orienting the determinant line bundle over moduli space

Let $\mathrm{V} \to B$ a vector bundle. Denote the i^{th} Stiefel-Whitney class of V by $w_i(\mathrm{V})$. Define two characteristic classes $p^{\pm}(\mathrm{V}) \in H^2(B, \mathbb{Z}/2\mathbb{Z})$ by

$$p^+(\mathrm{V}) = w_2(\mathrm{V}), \qquad p^-(\mathrm{V}) = w_2(\mathrm{V}) + w_1(\mathrm{V})^2. \tag{8.1}$$

From [KT] we know that $p^{\pm}(\mathrm{V})$ is the obstruction to the existence of a $Pin^{\pm}$ structure on V.

Definition 8.1 *Given a symplectic Lagrangian pair* (X, L)*, we say* L *is relatively* $Pin^{\pm}$ *if*

$$p^{\pm}(TL) \in \mathrm{Im}(i^* : H^2(X, \mathbb{Z}/2\mathbb{Z}) \to H^2(L, \mathbb{Z}/2\mathbb{Z})).$$

and is $Pin^{\pm}$ *if* $p^{\pm}(TL) = 0$*. If* L *is* $Pin^{\pm}$*, we define a* $Pin^{\pm}$ *structure for* L *to be a* $Pin^{\pm}$ *structure for* TL*. If* L *is relatively* $Pin^{\pm}$*, we define a relative* $Pin^{\pm}$ *structure for* (X, L) *consists of the choices of*

1. a triangulation for the pair (X, L),
2. an oriented vector bundle $\mathbb{V}$ over the three skeleton of X such that $w_2(\mathbb{V}) = p^{\pm}(TL)$,
3. a $Pin^{\pm}$ structure on $TL|_{L^{(3)}} \oplus \mathbb{V}|_{L^{(3)}}$.

Let us first introduce some notations. Recall that

$$\mathcal{M}^{V,\,\$}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d})$$

is the moduli space of (J,ν)-holomorphic V-regular open maps $u : (\Sigma, \partial\Sigma) \to (X, L)$ with k_a marked points $z_{a1}, \cdots, z_{ak_a}$ on each boundary component $(\partial\Sigma)_a$ and l marked points $w_1, \cdots, w_l$ and additional $\mathbb{l}$ interior intersecting marked points $q_1, \cdots, q_{\mathbb{l}}$ on Σ such that $u_*([\Sigma, \partial\Sigma]) = d$ and $u|_{(\partial\Sigma)_a*}([(\partial\Sigma)_a]) = d_a$, and $\ell = \mathbb{l} > 0$. The compactification $\overline{\mathcal{M}}^{V,\,\$}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d})$ is the space of V-stable open maps. We can define the evaluation maps at the $(\mathbf{k}, l)$-marked points and intersection points

$$evb_{ai} : \overline{\mathcal{M}}^{V,\,\$}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d}) \to L, \quad i = 1, \cdots, k_a, \; a = 1, \cdots, m,$$

$$evi_j : \overline{\mathcal{M}}^{V,\,\$}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d}) \to X, \quad j = 1, \cdots, l,$$

$$evi^I_{\jmath} : \overline{\mathcal{M}}^{V,\,\$}_{\mathbf{k},l,\mathbb{l}}(X, L, \mathbf{d}) \to V, \quad \jmath = 1, \cdots, \mathbb{l}.$$

In fact, the moduli space above can be considered as the zero locus of a Fredholm section of a Banach bundle. We denote by $B^{1,p}(X, L, \mathbf{d})$ the Banach manifold of $W^{1,p}$ maps $u : (\Sigma, \partial\Sigma) \to (X, L)$ such that $u_*([\Sigma, \partial\Sigma]) = d$ and $u|_{(\partial\Sigma)_a*}([(\partial\Sigma)_a]) = d_a$. And define

$$B^{1,p}_{\mathbf{k},l+\mathbb{l}}(X, L, \mathbf{d}) := B^{1,p}(X, L, \mathbf{d}) \times \prod_a (\partial\Sigma)_a^{k_a} \times \Sigma^{l+\mathbb{l}} \setminus \triangle,$$

where $\triangle$ denotes the subset of the product in which two marked points coincide. Elements of $B^{1,p}_{\mathbf{k},l+\mathbb{l}}(X, L, \mathbf{d})$ are denoted by $\mathbf{u} = (u, \vec{z}, \vec{w}, \vec{q})$, where $\vec{z} = (z_{ai})$, $\vec{w} = (w_j)$, $\vec{q} = (q_{\jmath})$.

Then the Banach space bundle $\mathcal{E} \to B^{1,p}_{\mathbf{k},l+\mathbb{l}}(X, L, \mathbf{d})$ is defined fiberwise with

$$\mathcal{E}_{\mathbf{u}} := L^p(\Sigma, \Omega^{0,1}(u^*TX)).$$

We define the section of this bundle as

$$\bar{\partial}_{(J,\nu)} : B^{1,p}_{\mathbf{k},l+\mathbb{l}}(X, L, \mathbf{d}) \to \mathcal{E}$$

$$\bar{\partial}_{(J,\nu)} u = du \circ j_\Sigma + J \circ du - \nu(\cdot, u(\cdot), \mathbf{u}),$$

which is the ν-perturbed Cauchy-Riemann operator. We require the inhomogeneous term

$$\nu \in \Gamma(\Sigma \times X \times B^{1,p}_{\mathbf{k},l+\mathbb{l}}(X, L, \mathbf{d}), \mathrm{Hom}(\pi_1^* T\Sigma, \pi_2^* TX)),$$

such that

(1) ν is (j_Σ, J)-anti-linear, *i.e.* $\nu \circ j_\Sigma = -J \circ \nu$;

(2) $\nu|_{\partial\Sigma \times X \times B^{1,p}_{\mathbf{k},l+\mathbb{l}}(X,L,\mathbf{d})}$ carries the sub-bundle $\pi'^*_1 T\partial\Sigma \subset \pi'^*_1 T\Sigma$ to the sub-bundle $\pi'^*_2(JTL) \subset \pi'^*_2 TX$, where π_i, $i = 1, 2$, is the projection from $\Sigma \times X \times B^{1,p}_{\mathbf{k},l+\mathbb{l}}(X, L, \mathbf{d})$ to the i^{th} factor.

The vertical component of the linearization of this section is denoted by

$$D := D\bar{\partial}_{(J,\nu)} : TB^{1,p}_{\mathbf{k},l+\mathbb{l}}(X, L, \mathbf{d}) \to \mathcal{E}.$$

Definition 8.2 *A $W^{1,p}$-map $u \in B^{1,p}_{\mathbf{k},l+\mathbb{l}}(X,L,\mathbf{d})$ is called V-regular if no component of its domain is mapped entirely into V and if neither any of the $|k|+l$ marked points nor any double points are mapped into V.*

Thus the parameterized V-regular moduli space can be regarded as the zero locus of the section $\bar{\partial}_{(J,\nu)}$ which is denoted by

$$\widetilde{\mathcal{M}}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\mathbf{d}) := \bar{\partial}^{-1}_{J,\nu}(0) \cap B^{1,p,\ V}_{\mathbf{k},l+\mathbb{l}}(X,L,\mathbf{d}). \tag{8.2}$$

Then similar to [So], we can define our moduli space $\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\mathbf{d})$ as an appropriate section (or say slice) of the reparameterization group action, we refer the reader to the section 4 in [So] for details of construction.

We simply denote by $B^V := B^{1,p,\ V}_{\mathbf{k},l+\mathbb{l}}(X,L,\mathbf{d})$. The evaluation maps can also be similarly defined on this larger space

$$evb_{ai} : B^V \to L, \quad i = 1,\cdots,k_a,\ a = 1,\cdots,m,$$

$$evi_j : B^V \to X, \quad j = 1,\cdots,l,$$

$$evi^I_\jmath : B^V \to V, \quad \jmath = 1,\cdots,\mathbb{l}, \qquad\qquad \text{if } \ \ell = \mathbb{l} \geq 1.$$

such that

$$evb_{ai}(\mathbf{u}) = u(z_{ai}), \quad evi_j(\mathbf{u}) = u(w_j), \quad evi^I_\jmath(\mathbf{u}) = u(q_\jmath).$$

The total evaluation map is denoted by

$$\mathbf{ev} := \prod_{a,i} evb_{ai} \times \prod_j evi_j \times \prod_{\mathbb{l}\geq 1,\jmath} evi^I_\jmath$$

$$\mathbf{ev} : B^{1,p,\ V}_{\mathbf{k},l+\mathbb{l}}(X,L,\mathbf{d}) \to L^{|\mathbf{k}|} \times X^l \times V^\ell. \tag{8.3}$$

Then let

$$\mathcal{L} := \det(D) \to B^{1,p,\ V}_{\mathbf{k},l+\mathbb{l}}(X,L,\mathbf{d}) \tag{8.4}$$

be the determinant line bundle of the family of Fredholm operators D restricted to B^V. And let

$$\mathcal{L}' = \mathcal{L} \otimes \bigotimes_{a,i} evb^*_{ai}\det(TL)^*.$$

The following two propositions are the fundamental results of orienting the moduli space. We distinguish the following two situations

(1) L is orientable and provided with an orientation. In this situation no restriction is required;

(2) L is nonorientable, then we suppose the number of marked points on each boundary component satisfies

$$k_a \cong w_1(d_a) + 1, \qquad (\mathrm{mod}\ 2); \tag{8.5}$$

Denote by $M^{(i)}$ the i-skeleton of a manifold M. Then we define a subspace

$$\mathfrak{B} := \{(u,\vec{z},\vec{w},\vec{q}) \in B^{1,p,\ V}_{\mathbf{k},l+\mathbb{l}}(X,L,\mathbf{d}) \mid u : (\Sigma,\partial\Sigma) \to (X^{(3)},L^{(3)})\}.$$

We first prove a lemma

Lemma 8.1 *The combination of the relative* $Pin^{\pm}$ *structure of* (X, L) *and the orientations of* L *if it is orientable determines a canonical orientation of* $\mathcal{L}'|_{\mathfrak{B}}$.

Proof. We just need canonically orient each individual line $\mathcal{L}'_{\mathbf{u}}$ for each $\mathbf{u} \in \mathfrak{B}$ in a way that varies continuously in families. Recall that the relative $Pin^{\pm}$ structure of (X, L) gives a vector bundle $\mathbb{V} \to X^{(3)}$ and a $Pin^{\pm}$ structure on $\mathbb{V}|_{L^{(3)}} \oplus TL|_{L^{(3)}}$. Denote simply by

$$\mathbb{V}_{\mathbb{R}} := \mathbb{V}|_{L^{(3)}}, \qquad \mathbb{V}_{\mathbb{C}} := \mathbb{V} \otimes \mathbb{C}. \tag{8.6}$$

Choosing an arbitrary Cauchy-Riemann operator D_0 on $u^*\mathbb{V}_{\mathbb{C}}$, we consider the operator $D_{\mathbf{u}} \oplus D_0$,

$$D_{\mathbf{u}} \oplus D_0 : TB_{\mathbf{u}} \oplus W^{1,p}(u^*\mathbb{V}_{\mathbb{C}}, u^*\mathbb{V}_{\mathbb{R}}) \longrightarrow \mathcal{E} \oplus L^p(\Omega^{0,1}(u^*\mathbb{V}_{\mathbb{C}})),$$

That is

$$W^{1,p}_{\mathbf{s}}(u^*(TX \oplus \mathbb{V}_{\mathbb{C}}), u|^*_{\partial\Sigma}(TL \oplus \mathbb{V}_{\mathbb{R}})) \oplus \mathbb{R}^{|\mathbf{k}|} \oplus \mathbb{C}^{l+\ell} \to L^p(\Omega^{0,1}(u^*(TX \oplus \mathbb{V}_{\mathbb{C}}))).$$

We remark that the choice of D_0 is irrelevant since the space of real linear Cauchy-Riemann operators on $u^*\mathbb{V}_{\mathbb{C}}$ is contractible.

Thus, we have a short exact sequence of Fredholm operators

$$0 \to D_{\mathbf{u}} \to D_{\mathbf{u}} \oplus D_0 \to D_0 \to 0.$$

By Lemma A.1 there exists an isomorphism

$$\det(D_{\mathbf{u}}) \simeq \det(D_{\mathbf{u}} \oplus D_0) \otimes \det(D_0)^*.$$

Tensor by $\bigotimes_{a,i}(evb^*_{ai}\det(TL)^*)_{\mathbf{u}}$ on both sides,

$$\begin{aligned} \mathcal{L}'_{\mathbf{u}} &= \det(D_{\mathbf{u}}) \otimes \bigotimes_{a,i}(evb^*_{ai}\det(TL)^*)_{\mathbf{u}} \\ &\simeq \det(D_{\mathbf{u}} \oplus D_0) \otimes \det(D_0)^* \otimes \bigotimes_{a,i}(evb^*_{ai}\det(TL)^*)_{\mathbf{u}}. \end{aligned} \tag{8.7}$$

Actually, we only need to orient

$$\mathcal{L}'_{\mathbf{u}} \otimes \det(D_0) \simeq \det(D_{\mathbf{u}} \oplus D_0) \otimes \bigotimes_{a,i}(evb^*_{ai}\det(TL)^*)_{\mathbf{u}}, \tag{8.8}$$

since the Cauchy-Riemann Pin boundary value problem

$$\underline{D}_0 = (\Sigma, u^*\mathbb{V}_{\mathbb{C}}, u^*\mathbb{V}_{\mathbb{R}}, \mathfrak{P}_0, D_0)$$

determines a canonical orientation on $\det(D_0)$. Indeed, note that any real vector bundle over a Riemann surface Σ with nonempty boundary admits a Pin structure because the second cohomology $H^2(\Sigma)$ is trivial. So we can choose a Pin structure $\widetilde{\mathfrak{P}}_0$ on $u^*\mathbb{V} \to \Sigma$ and define $\mathfrak{P}_0$ to be its restriction to $u^*\mathbb{V}_{\mathbb{R}} \to \partial\Sigma$. By the Lemma 2.11 in [So], the canonical orientation on $\det(D_0)$ does not depend on the choice of $\widetilde{\mathfrak{P}}_0$.

Recall that the relative Pin structure on (X, L) gives a Pin structure on $TL|_{L^{(3)}} \oplus \mathbb{V}|_{L^{(3)}}$, and so gives a Pin structure on $u|_{\partial\Sigma}^{*}(TL \oplus \mathbb{V}_{\mathbb{R}})$. The notation is simplified since $u \in \mathfrak{B}$.

If L is orientable and given an orientation, we have induced orientations on $u|_{\partial\Sigma}^{*}(TL \oplus \mathbb{V}_{\mathbb{R}})$. Thus we consider the restricted Pin boundary value problem $\underline{D_{\mathbf{u}} \oplus D_0}$, the Proposition A.1 and the Remark after it ensures there exists a canonical orientation on $\det(D_{\mathbf{u}} \oplus D_0)$. Then the orientation of L gives the orientation of $\det(TL)$. Therefore, we provide a canonical orientation on the right-hand side of (8.8).

If L is non-orientable, on each boundary component $(\partial\Sigma)_a$ such that $k_a \neq 0$, choose arbitrarily an orientation on $(evb_{a1}^{*}TL)_{\mathbf{u}}$. Still note that each boundary component $(\partial\Sigma)_a$ has an orientation induced from the natural orientation on Σ. For each $i \in [2, k_a]$ if $k_a \neq 0$, we can obtain an orientation on $(evb_{ai}^{*}TL)_{\mathbf{u}}$ by trivializing $u|_{\partial\Sigma}^{*}TL$ along the oriented line segment in $(\partial\Sigma)_a$ from z_{a1} to z_{ai}. If for some a, $u|_{(\partial\Sigma)_a}^{*}TL$ is orientable, and if $k_a \neq 0$, then the orientation on $(evb_{a1}^{*}TL)_{\mathbf{u}}$ induces an orientation on $u|_{(\partial\Sigma)_a}^{*}TL$; otherwise, if $k_a = 0$, then we arbitrarily choose an orientation on $u|_{(\partial\Sigma)_a}^{*}TL$. Thus for such a, $u|_{(\partial\Sigma)_a}^{*}(TL \oplus \mathbb{V}_{\mathbb{R}})$ also has an orientation by choosing an orientation on $\mathbb{V}_{\mathbb{R}}$. Similar to the above argument $u|_{\partial\Sigma}^{*}(TL \oplus \mathbb{V}_{\mathbb{R}})$ admits a Pin structure given by the relative Pin structure on (X, L). Then the Proposition A.1 and the Remark after it still apply to $\underline{D_{\mathbf{u}} \oplus D_0}$, and $\det(D_{\mathbf{u}} \oplus D_0)$ has a canonical orientation. Hence we also provide a canonical orientation on the right-hand side of (8.8). Note that under the additional assumption (8.5), the choice of orientation on $(evb_{a1}^{*}TL)_{\mathbf{u}}$ is not important. Because changing the orientation on $(evb_{a1}^{*}TL)_{\mathbf{u}}$ will change all the orientations on $u|_{(\partial\Sigma)_a}^{*}(TL \oplus \mathbb{V}_{\mathbb{R}})$ and on $(evb_{ai}^{*}TL)_{\mathbf{u}}$, $i \in [2, k_a]$ it induces, by Proposition A.1 and Remark after it, the orientation on $\det(D_{\mathbf{u}} \oplus D_0)$ will also change. Then the assumption (8.5) ensures that the number of orientation changes is even. So the orientation on the right-hand side of (8.8) is invariant.

Note that the construction above varies continuously in a one-parameter family, we thus canonically oriented $\mathcal{L}'|_{\mathfrak{B}}$. □

Proposition 8.1 *Under the assumptions in Lemma 8.1, the combination of the orientations of L if it is orientable and the choice of relative $Pin^{\pm}$ structure $\mathfrak{P}$ on L provides a canonical orientation on $\mathcal{L}'$, that is to say, there exists a canonical isomorphism of line bundles*

$$\mathcal{L} \xrightarrow{\sim} \bigotimes_{a,i} evb_{ai}^{*}\det(TL). \tag{8.9}$$

Proof. It suffices to provide a canonical orientation for the fiber $\mathcal{L}'_{\mathbf{u}}$ over each $\mathbf{u} \in B^V = B_{\mathbf{k}, l+\mathbb{1}}^{1,p,\,V}(X, L, \mathbf{d})$ such that it varies continuously with $\mathbf{u}$.

By definition the relative $Pin^{\pm}$ structure gives a triangulation of the pair (X, L). The map $u : (\Sigma, \partial\Sigma) \to (X, L)$ is homotopic to a map $\hat{u} : (\Sigma, \partial\Sigma) \to (X^{(2)}, L^{(2)})$ by using simplicial approximation. The homotopy map is denoted by

$$\Phi : [0, 1] \times (\Sigma, \partial\Sigma) \to (X, L).$$

We will show that the choice of map Φ is unique up to homotopy. Indeed, let Φ' be another such map. We can get a new map by concatenating Φ and Φ'

$$\Phi\#\Phi' : [-1, 1] \times (\Sigma, \partial\Sigma) \to (X, L).$$

We can use simplicial approximation again to homotope $\Phi\#\Phi'$ to a map into the three skeleton $(X^{(3)}, L^{(3)})$. By retaking suitable parameters, we get a homotopy map from Φ to Φ', denote it by

$$\Psi : [0,1]^2 \times (\Sigma, \partial\Sigma) \to (X, L),$$

satisfying

$$\Psi(0,t) = \Phi(t), \qquad \Psi(1,t) = \Phi'(t), \qquad \Psi(s,0) = u.$$

That is to say Φ is unique up to homotopy.

On the other hand, the two maps Φ and Ψ can be considered as maps from $[0,1]$ and $[0,1]^2$ to B^V, respectively. Note that $\hat{u} \in \mathfrak{B}$, by the Lemma 8.1, the relative $Pin^{\pm}$ structure of (X, L) determines a canonical orientation of $\mathcal{L}'|_{\mathfrak{B}}$. Then the orientation on $\mathcal{L}'|_{\hat{\mathbf{u}}}$ induces an orientation of $\mathcal{L}'|_{\mathbf{u}}$ by trivializing $\Phi^*\mathcal{L}'$. Such orientation is the same as the one induced by any other homotopy Φ' since we can trivialize $\Psi^*\mathcal{L}'$. It is easy to see such induced orientation on $\mathcal{L}'|_{\mathbf{u}}$ varies continuously with $\mathbf{u}$. Therefore, $\mathcal{L}'$ admits a canonical orientation. $\square$

The isomorphism (8.9) doesn't involve the effect of the ordering of the marked points on boundary components. In fact, $B^{1,p,\ V}_{\mathbf{k},l+\mathbb{1}}(X, L, \mathbf{d})$ consists many connected components, each one corresponds to each ordering of boundary marked points. Denote

$$\varpi = (\varpi_1, \cdots, \varpi_m)$$

where each ϖ_a is a permutation of the integers $1, \cdots, k_a$. We define the sign of ϖ

$$\operatorname{sign}(\varpi) := \sum_a \operatorname{sign}(\varpi_a). \tag{8.10}$$

Denote by B^V_{ϖ} the component of $B^{1,p,\ V}_{\mathbf{k},l+\mathbb{1}}(X, L, \mathbf{d})$ in which $\vec{z}$ are ordered in $\partial\Sigma$ by ϖ. Now we modify the canonical isomorphism in Proposition 8.1 by

Definition 8.3 *When* $\dim L \simeq 0 \pmod 2$ *we define the canonical isomorphism to be the isomorphism constructed in the Proposition 8.1 twisted by* $(-1)^{\operatorname{sign}(\varpi)}$ *over the component* $B^{1,p}_{\mathbf{k},l,\varpi}(\Sigma, L, \mathbf{d})$. *Otherwise, we define the isomorphism to be exactly the isomorphism constructed in the Proposition 8.1.*

Now we consider the orientation of moduli space of V-stable maps. We will restrict attention to the special case: V-stable map of two components, one of which is from the original Riemann surface Σ, and the other of which is a disc bubble, no component is mapped into V;

Recall that the domain is equipped with a marked layer structure (see Definition 7.2). Similar to (7.1), we can define associated operators D^N_K on the layers Λ_K, $K \geq 1$.

Definition 8.4 *A* V*-stable* $W^{1,p}$*-map is a map* $u \in B^{1,p,\ V}_{\mathbf{k},l+\mathbb{1}}(\Sigma, X, L, \mathbf{d})$ *together with*

(i) *a marked layer structure on its (nodal) domain* Σ *with* $u|_{\Lambda_0}$ *being* V*-regular, and*

(ii) *for each* $K \geq 1$, *an element* $\xi_K \in \ker D^N_K$ *defined on the layer* Λ_K *that is a section nontrivial on every irreducible component of* Λ_K.

In particular, denote the space of V-stable $W^{1,p}$-maps with only the layer $K=0$ components by $B^{1,p,\ V}_{\mathbf{k},l,\mathbb{l}}(\Sigma, X, L, \mathbf{d}, 0)$, and denote the space of V-stable $W^{1,p}$-maps with only the layer $K=1$ components by $B^{1,p,\ V}_{\mathbf{k},l,\mathbb{l}}(\Sigma, X, L, \mathbf{d}, 1)$.

We will give a description in more detail. Suppose that only one disc bubbles off the boundary component $(\partial\Sigma)_b$ along with k'' of the marked points on $(\partial\Sigma)_b$ and l'' of the interior marked points. The domain is a nodal surface

$$\hat{\Sigma} = \Sigma \cup D^2 / z_0' \sim z_0''$$

with $|\mathbf{k}'|+1$ boundary marked points and $l'+\mathbb{l}'$ interior marked points on Σ, and $k''+1$ boundary marked points and $l''+\mathbb{l}''$ interior marked points on D^2 such that

$$\mathbf{k}' = (k_1, \cdots, k', \cdots, k_m), \quad k' = k_b - k'', \quad l = l' + l'', \quad \mathbb{l} = \mathbb{l}' + \mathbb{l}''.$$

We denote by z_0' (resp. z_0'') the extra marked point on Σ (resp. D^2) which is different from any z_{bi}, and also denote $\ell' = \mathbb{l}'$, $\ell'' = \mathbb{l}''$. Denote by

$$\begin{aligned} B^{V\#} &= B^{1,p,\ V}_{\mathbf{k},\sigma,l,\rho,\mathbb{l},\varrho}(\hat{\Sigma}, X, L, \mathbf{d}', d'', 0) \\ &:= B^{1,p,\ V}_{\mathbf{k}'+e_b,l',\mathbb{l}'}(\Sigma, X, L, \mathbf{d}', 0)_{evb_0'} \times_{evb_0''} B^{1,p,\ V}_{k''+1,l'',\mathbb{l}''}(D^2, X, L, d'', 0) \end{aligned}$$

the space of V-regular $W^{1,p}$ stable maps with only one disc bubbling off and no component is mapped into V as above. The disc bubble represents the class $d'' \in H_2(X, L)$, denote

$$\mathbf{d}' = (d', d_1, \cdots, d_b', \cdots, d_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m},$$

satisfying $d' + d'' = d$, $d_b' + \partial d'' = d_b$, and $e_b = (0, \cdots, 1, \cdots, 0)$ is the vector with only the b^{th} element equal to 1, others are zeros.

In the notation of the $W^{1,p}$ space above, $\sigma \subset [1, k_b]$ and $\rho \subset [1, l]$ denote the subsets of boundary and interior bubble off marked points, respectively. And $\varrho \subset [1, \mathbb{l}]$ denote the subsets of intersection interior bubble off marked points.

We write the element $\mathbf{u} = (\mathbf{u}', \mathbf{u}'') \in B^{1,p,\ V}_{\mathbf{k},\sigma,l,\rho,\mathbb{l},\varrho}(\hat{\Sigma}, X, L, \mathbf{d}', d'', 0)$, where

$$\mathbf{u}' \in B' := B^{1,p,\ V}_{\mathbf{k}'+e_b,l',\mathbb{l}'}(\Sigma, X, L, \mathbf{d}', 0),$$

$$\mathbf{u}'' \in B'' := B^{1,p,\ V}_{k''+1,l'',\mathbb{l}''}(D^2, X, L, d'', 0)$$

such that

$$evb_0'(\mathbf{u}') = evb_0''(\mathbf{u}''),$$

where evb_0' (resp. evb_0'') is the evaluation map at z_0' (resp. z_0''). For each such $\mathbf{u} \in B^{V\#}$, we denote by

$$\hat{\Sigma}_{\mathbf{u}} := \Sigma \cup D^2 / z_0' \sim z_0''.$$

the associated domain curve. The stable map is $u : (\hat{\Sigma}, \partial\hat{\Sigma}) \to (X, L)$, and the node of $\hat{\Sigma}$ is denoted by z_0.

If we denote the two natural projections by

$$p_1 : B^{V\#} \to B', \qquad p_2 : B^{V\#} \to B'',$$

then we can define the Banach bundle $\mathcal{E}^{\#} \to B^{V\#}$ by

$$\mathcal{E}^{\#} := p_1^*\mathcal{E}' \oplus p_2^*\mathcal{E}''.$$

with fiber

$$\mathcal{E}^{\#}_{\mathbf{u}} := L^p(\hat{\Sigma}_{\mathbf{u}},\ \Omega^{0,1}(u'^*TX) \oplus \Omega^{0,1}(u''^*TX)).$$

For $(J,\nu) \in \mathbb{J}^V$, we denote by

$$\bar{\partial}^{\#}_{(J,\nu)} : B^{V\#} \to \mathcal{E}^{\#}$$

the section given by the ν-perturbed Cauchy-Riemann operator. The vanishing set of this section is the parameterized one-disc-bubble moduli space, denoted by

$$(\bar{\partial}^{\#}_{(J,\nu)})^{-1}(0) = \widetilde{\mathcal{M}}^{V,\ \mathbb{s}}_{\mathbf{k},\sigma,l,\rho,\mathbb{I},\varrho}(\hat{\Sigma}, X, L, \mathbf{d}', d'', 0). \tag{8.11}$$

Its vertical linearization is

$$D^{\#} := D\bar{\partial}^{\#}_{(J,\nu)} : TB^{V\#} \to \mathcal{E}^{\#}.$$

Denote the determinant line bundle of family of Fredholm operators $D^{\#}$ by

$$\mathcal{L}^{\#} := \det(D^{\#}) \to B^{V\#}.$$

Denote

$$\mathcal{L}^{\#'} = \mathcal{L}^{\#} \otimes \bigotimes_{a,i}(evb_{ai}^*\det(TL)^*.$$

We define a subspace

$$\mathfrak{B}^{\#} := \{(u, \vec{z}, \vec{w}, \vec{q}) \in B^{V\#} \mid u : (\hat{\Sigma}, \partial\hat{\Sigma}) \to (X^{(3)}, L^{(3)})\}.$$

We can obtain a lemma similar to the Lemma 8.1

Lemma 8.2 *The combination of the relative $Pin^{\pm}$ structure of (X, L) and the orientation of L if it is orientable determines a canonical orientation of $\mathcal{L}^{\#'}|_{\mathfrak{B}^{\#}}$.*

Proof. It is similar to the proof of Lemma 8.1 and the arguments in Proposition 3.3 of [So], the modifications take place when we apply the Proposition A.1 and the Remark after it to the two restricted Pin boundary value problems

$$\underline{D'_{\mathbf{u}} \oplus D'_0} \quad \text{and} \quad \underline{D''_{\mathbf{u}} \oplus D''_0}.$$

And we have an isomorphism

$$\det(D^{\#}_{\mathbf{u}} \oplus D^{\#}_0) \simeq \det(D'_{\mathbf{u}} \oplus D'_0) \otimes \det(D''_{\mathbf{u}} \oplus D''_0) \otimes evb_0^*\det(TL \oplus V_{\mathbb{R}})^*_{\mathbf{u}}. \tag{8.12}$$

$\square$

Using the same argument of homotopy uniqueness in the proof of Proposition 8.1, we can similarly obtain the following

Proposition 8.2 *The combination of the orientations of L if it is orientable and the choice of relative* $Pin^{\pm}$ *structure* $\mathfrak{P}$ *on* L *provide a canonical orientation on* $\mathcal{L}^{\#'}$*, that is to say, there exists a canonical isomorphism of line bundles*

$$\mathcal{L}^{\#} \xrightarrow{\sim} \bigotimes_{a,i} evb_{ai}^{*}\det(TL). \tag{8.13}$$

In order to involve the effect of the ordering of the marked points, we also need modify the isomorphism in the Proposition 8.2. Recall from the proof above that the marked points on each $(\partial\hat{\Sigma})_a$ can be canonically ordered. So we can divide the space $B^{1,p}_{\mathbf{k},l}(\Sigma, L, \mathbf{d})$ into components $B^{1,p}_{\mathbf{k},l,\varpi}(\Sigma, L, \mathbf{d})$. To be consistent with the isomorphism defined in Definition 8.3, we can modify the canonical isomorphism (8.13) and have the following definition.

Definition 8.5 *When* $\dim L \simeq 0 \pmod 2$ *we define the canonical isomorphism to be the isomorphism constructed in the Proposition 8.2 twisted by* $(-1)^{\mathrm{sign}(\varpi)}$ *over the component* $B^{1,p}_{\mathbf{k},l,\varpi}(\Sigma, L, \mathbf{d})$*. Otherwise, we define the isomorphism to be exactly the isomorphism constructed in the Proposition 8.2.*

9 Involution and sign

Here the argument is different from the discussion by Ionel-Parker [IP1] for closed relative GW-invariant, we must deal with the codimension 1 frontier of the compactification of the moduli space of the open V-regular maps. We will modify the original method by J. Solomon [So]. Note in the sequel we still assume $L \cap V = \emptyset$.

Let us then suppose there exists an anti-symplectic involution ϕ such that $L = \mathrm{Fix}(\phi)$. And suppose Σ is biholomorphic to its conjugation $\bar{\Sigma}$, *i.e.* there exists an anti-holomorphic involution $c : \Sigma \to \Sigma$. Fix $(J, \nu) \in \mathbb{J}^V_{\phi}$. Thus, from the V-regular (J, ν)-holomorphic (resp. $W^{1,p}$) map $u : (\Sigma, \partial\Sigma) \mapsto (X, L)$ we can define its conjugate (J, ν)-holomorphic (resp. $W^{1,p}$) map $\tilde{u} = \phi \circ u \circ c$ representing the homology class $\tilde{d} = [\tilde{u}]$. So we have an induced map

$$\phi' : B^{1,p,\ V}_{\mathbf{k},l+\mathbb{1}}(X, L, \mathbf{d}) \to B^{1,p,\ V}_{\mathbf{k},l+\mathbb{1}}(X, L, \tilde{\mathbf{d}})$$

given by

$$\mathbf{u} = (u, \vec{z}, \vec{w}, \vec{q}) \mapsto \tilde{\mathbf{u}} = (\tilde{u},\ (c|_{\partial\Sigma})^{|\mathbf{k}|}(\vec{z}),\ c^{l}(\vec{w}),\ c^{\ell}(\vec{q})).$$

We denote the relevant Banach space bundle by $\widetilde{\mathcal{E}} \to B^{1,p,\ V}_{\mathbf{k},l+\mathbb{1}}(X, L, \tilde{\mathbf{d}})$ with fiber

$$\widetilde{\mathcal{E}}_{\tilde{\mathbf{u}}} := L^p(\Sigma, \Omega^{0,1}(\tilde{u}^{*}TX)).$$

And we can similarly get a determinant line bundle of a family of Fredholm operators $\widetilde{D} : TB^{1,p,\ V}_{\mathbf{k},l+\mathbb{1}}(X, L, \tilde{\mathbf{d}}) \to \widetilde{\mathcal{E}}$ as

$$\widetilde{\mathcal{L}} := \det(\widetilde{D}) \to B^{1,p,\ V}_{\mathbf{k},l+\mathbb{1}}(X, L, \tilde{\mathbf{d}}).$$

The evaluation maps can also be similarly defined

$$\widetilde{ev}b_{ai} : B^{1,p,\ V}_{\mathbf{k},l+\mathbb{l}}(X, L, \tilde{\mathbf{d}}) \to L, \quad i = 1, \cdots, k_a,\ a = 1, \cdots, m,$$

$$\widetilde{ev}i_j : B^{1,p,\ V}_{\mathbf{k},l+\mathbb{l}}(X, L, \tilde{\mathbf{d}}) \to X, \quad j = 1, \cdots, l,$$

$$\widetilde{ev}i^I_\jmath : B^{1,p,\ V}_{\mathbf{k},l+\mathbb{l}}(X, L, \tilde{\mathbf{d}}) \to V, \quad \jmath = 1, \cdots, \mathbb{l},$$

by

$$\widetilde{ev}b_{ai}(\tilde{\mathbf{u}}) = \tilde{u}(c(z_{ai})), \quad \widetilde{ev}i_j(\tilde{\mathbf{u}}) = \tilde{u}(c(w_j)), \quad \widetilde{ev}i^I_\jmath(\tilde{\mathbf{u}}) = \tilde{u}(c(q_\jmath)).$$

The total evaluation map is denoted by

$$\widetilde{\mathbf{ev}} := \prod_{a,i} \widetilde{ev}b_{ai} \times \prod_j \widetilde{ev}i_j \times \prod_\jmath \widetilde{ev}i^I_\jmath$$

$$\widetilde{\mathbf{ev}} : B^{1,p,\ V}_{\mathbf{k},l+\mathbb{l}}(X, L, \tilde{\mathbf{d}}) \to L^{|\mathbf{k}|} \times X^l \times V^\ell. \tag{9.1}$$

Moreover, we can define a map

$$\Phi : \mathcal{E} \to \widetilde{\mathcal{E}}$$

$$\xi \mapsto d\phi \circ \xi \circ dc$$

covering ϕ'. Also Φ induces a map $\Psi : \mathcal{L} \to \widetilde{\mathcal{L}}$ covering ϕ'. Denote by

$$\widetilde{\mathcal{L}}' := \widetilde{\mathcal{L}} \otimes \bigotimes_{a,i} \widetilde{ev}b^*_{ai} \det(TL)^*.$$

Thus, Ψ also induces a map

$$\Psi' : \mathcal{L}' \to \widetilde{\mathcal{L}}'$$

covering ϕ'.

From the Proposition 8.1 and Definition 8.3 we know that both $\mathcal{L}'$ and $\widetilde{\mathcal{L}}'$ have canonical orientation. So the map Ψ' may either preserve the orientation component or reverse the orientation of some connected components. We say the sign of Ψ' is 0 if Ψ' preserves the orientation of $\mathcal{L}'$ to that of $\widetilde{\mathcal{L}}'$, otherwise, we say the sign of Ψ' is 1. The following proposition shows an expression of the sign of Ψ'. Recall g_0 denotes the genus of $\Sigma/\partial\Sigma$, and $\dim L = n$.

Proposition 9.1 *If L is relative Pin^-, then the map Ψ' has sign*

$$\begin{aligned} \mathfrak{s}^- \quad \cong \quad & \frac{\mu(d)(\mu(d)+1)}{2} + (1-g_0)n + nm + \deg \mathfrak{s} + |\mathbf{k}| + l + \ell \\ & + w_2(\mathbb{V})(\psi(d)) + w_1(u^*TL)(\partial d) + \sum_{a<b} w_1(u^*TL)(d_a) w_1(u^*TL)(d_b) \\ & + \sum_a w_1(u^*TL)(d_a)(k_a - 1) + (n+1) \sum_a \frac{(k_a-1)(k_a-2)}{2} \qquad \mod 2. \end{aligned}$$

If L is relative Pin^+, then the map Ψ' has sign

$$\begin{aligned}\mathfrak{s}^- &\cong \frac{\mu(d)(\mu(d)+1)}{2} + (1-g_0)n + nm + \deg \mathbb{s} + |\mathbf{k}| + l + \ell \\ &+ w_2(\mathbb{V})(\psi(d)) + \sum_{a<b} w_1(u^*TL)(d_a)w_1(u^*TL)(d_b) \\ &+ \sum_a w_1(u^*TL)(d_a)(k_a-1) + (n+1)\sum_a \frac{(k_a-1)(k_a-2)}{2} \mod 2.\end{aligned}$$

Where $\psi : H_*(X, L; \mathbb{Z}/2\mathbb{Z}) \to H_*(X; \mathbb{Z}/2\mathbb{Z})$ is a degree 0 homomorphism defined in [So].

Since the proof of the preceding proposition is similar to the proof of the next one and we will not actually use it, we do not show the proof here.

Sometimes, for simplicity, we would denote $\mu = \mu(d)$, $w_2 = w_2(\mathbb{V})$ and $w_1 = w_1(u^*TL)$ if no danger of confusion.

Corollary 1 *If* $\dim L \leq 3$, *then we always have*

$$\begin{aligned}\mathfrak{s}^- &\cong \frac{\mu(\mu+1)}{2} + (1-g_0)n + nm + \deg \mathbb{s} + |\mathbf{k}| + l + \ell \\ &+ \sum_{a<b} w_1(d_a)w_1(d_b) + \sum_a w_1(d_a)(k_a-1) \\ &+ (n+1)\sum_a \frac{(k_a-1)(k_a-2)}{2} \mod 2. \end{aligned} \tag{9.2}$$

In particular, if $\Sigma = D^2$, we have

$$\begin{aligned}\mathfrak{s}^- &\cong \frac{\mu(\mu-1)}{2} + \deg \mathbb{s} + |\mathbf{k}| + l + \ell + \mu(k-1) \\ &+ (n+1)\frac{(k-1)(k-2)}{2} \mod 2. \end{aligned} \tag{9.3}$$

Proof. When $\dim L \leq 3$, the Wu relations imply that L is Pin^-, we can take the standard Pin^- structure. Note that $w_2(\mathbb{V}) = 0$ and $w_1(\partial d) \cong \mu(d) \mod 2$, thus the formula is simplified. $\square$

Then we define and calculate the sign of a map related to the boundary of moduli space. Recall that

$$\begin{aligned} B^{V\#} &= B^{1,p,\ V}_{\mathbf{k},\sigma,l,\rho,\mathbb{I},\varrho}(\hat{\Sigma}, X, L, \mathbf{d}', d'', 0) \\ &:= B^{1,p,\ V}_{\mathbf{k}'+e_b,l',\mathbb{I}'}(\Sigma, X, L, \mathbf{d}', 0)_{evb'_0} \times_{evb''_0} B^{1,p,\ V}_{k''+1,l'',\mathbb{I}''}(D^2, X, L, d'', 0)\end{aligned}$$

Note that the standard conjugation $c : D^2 \to D^2$ gives a biholomorphic isomorphism $D^2 \simeq \bar{D}^2$. Then from the involution ϕ, we just have an induced map

$$\phi_{B''} : B^{1,p,\ V}_{k''+1,l'',\mathbb{I}''}(D^2, X, L, d'', 0) \to B^{1,p,\ V}_{k''+1,l'',\mathbb{I}''}(D^2, X, L, \widetilde{d''}, 0)$$

of the second factor of the fiber product. Then since $L = \mathrm{Fix}(\phi)$, $\phi_{B''}$ induces an map of the whole fiber product

$$\phi_{B\#} : B^{V\#} \to \widetilde{B}^{V\#}$$

$$B^{1,p,\ V}_{\mathbf{k},\sigma,l,\rho,\mathbb{1},\varrho}(\hat{\Sigma}, X, L, \mathbf{d}', d'', 0) \to B^{1,p,\ V}_{\mathbf{k},\sigma,l,\rho,\mathbb{1},\varrho}(\hat{\Sigma}, X, L, \mathbf{d}', \widetilde{d''}, 0).$$

We denote the relevant Banach space bundle by

$$\mathcal{E}^{\#} \to B^{V\#}, \qquad \widetilde{\mathcal{E}}^{\#} \to \widetilde{B}^{V\#}.$$

And we can similarly get a determinant line bundle of a family of Fredholm operators $\widetilde{D} : TB^{V\#} \to \widetilde{\mathcal{E}}^{\#}$ as

$$\widetilde{\mathcal{L}}^{\#} := \det(\widetilde{D}) \to B^{1,p,\ V}_{\mathbf{k},\sigma,l,\rho,\mathbb{1},\varrho}(\hat{\Sigma}, X, L, \mathbf{d}', \widetilde{d''}, 0).$$

The obvious evaluation maps are denoted by $\widetilde{evi}_j$, $\widetilde{evb}_{ai}$, $\widetilde{evi}^I_j$. Denote by

$$\widetilde{\mathcal{L}}^{\#'} := \widetilde{\mathcal{L}}^{\#} \otimes \bigotimes_{a,i} \widetilde{evb}^*_{ai} \det(TL)^*.$$

Similarly, we have induced map $\Phi^{\#} : \mathcal{E}^{\#} \to \widetilde{\mathcal{E}}^{\#}$ covering $\phi_{B^{\#}}$. Recall that the inhomogeneous term ν vanishes on bubble components, so it is ϕ-invariant. Therefore, $\bar{\partial}^{\#}_{(J,\nu)}|_{\mathcal{E}^{\#}}$ and $\bar{\partial}^{\#}_{(J,\nu)}|_{\widetilde{\mathcal{E}}^{\#}}$ are two $\phi_{B^{\#}} - \Phi^{\#}$ equivariant sections. Consequently, $\phi_{B^{\#}}$ and $\Phi^{\#}$ induce a map of determinant line bundles $\Psi^{\#} : \mathcal{L}^{\#} \to \widetilde{\mathcal{L}}^{\#}$ covering $\phi_{B^{\#}}$. Thus, $\Psi^{\#}$ also induces a map

$$\Psi^{\#'} : \mathcal{L}^{\#'} \to \widetilde{\mathcal{L}}^{\#'}$$

covering $\phi_{B^{\#}}$. If this map preserves orientation the sign of it is 0, otherwise the sign is 1.

Remark. If the homology class d represented by u is ϕ-anti-invariant, that is, the stable map u is real, then all the maps above are involutions of their respective objects. In particular, we only define and calculate the sign of $\Psi^{\#'} : \mathcal{L}^{\#} \to \mathcal{L}^{\#}$.

For stating the formulae for the sign of $\Psi^{\#'}$, we introduce some new notation. Let

$$\Upsilon^{(1)}(d'', k'') :\cong \mu(d'')k'' \cong w_1(\partial d'')k'', \tag{9.4}$$

$$\Upsilon^{(2)}(d'_b, d'', k', k'') :\cong \begin{cases} 0, & w_1(d'_b) = w_1(\partial d'') = 0, \\ k', & w_1(d'_b) = w_1(\partial d'') = 1, \\ k'' - 1, & w_1(d'_b) = 1,\ w_1(\partial d'') = 0, \\ k' + k'' - 1, & w_1(d'_b) = 0,\ w_1(\partial d'') = 1. \end{cases} \tag{9.5}$$

Proposition 9.2 *(1) Suppose the marked point z_{b1} does not bubble off, i.e. $1 \notin \sigma$. Then the map $\Psi^{\#'}$ has sign*

$$\begin{aligned} \mathfrak{s}^{\#(1)}_{\pm} \quad \cong \quad & \frac{\mu(d'')(\mu(d'') \pm 1)}{2} + \deg \mathbb{s}'' + k'' + 1 + l'' + \ell'' \\ & + w_2(\mathbb{V})(\psi(d'')) + \Upsilon^{(1)}(d'', k'') \\ & + (n+1)\frac{k''(k''-1)}{2} \qquad \mod\ 2, \end{aligned} \tag{9.6}$$

with + in the Pin^+ and − in the Pin^- case.

(2) Suppose the marked point z_{b1} bubbles off, i.e. $1 \in \sigma$. Then the map $\Psi^{\#'}$ has sign

$$\begin{aligned}
\mathfrak{s}_{\pm}^{\#(2)} \quad \cong \quad & \frac{\mu(d'')(\mu(d'') \pm 1)}{2} + \deg \mathfrak{s}'' + k'' + 1 + l'' + \ell'' \\
& + w_2(\mathbb{V})(\psi(d'')) + \Upsilon^{(2)}(d_b', d'', k', k'') + w_1(d_b')w_1(\partial d'') \\
& + (n+1)[\frac{(k''-1)(k''-2)}{2} + k_b(k''+1)] \quad \mod 2,
\end{aligned} \tag{9.7}$$

with + in the Pin^+ and $-$ in the Pin^- case.

Remark. If L is orientable and $\dim L = n$ is odd. Note the fact that if L is orientable then $\mu(d'')$ is even , so $w_1(d_b') = w_1(\partial d'') = 0$, we have

$$\begin{aligned}
\mathfrak{s}^{\#(1)} \quad = \quad & \mathfrak{s}^{\#(2)} = \mathfrak{s}^{\#(3)} = \mathfrak{s}^{\#(4)} \\
\cong \quad & \frac{\mu(d'')}{2} + \deg \mathfrak{s}'' + k'' + 1 + l'' + \ell'' + w_2(\mathbb{V})(\psi(d'')).
\end{aligned} \tag{9.8}$$

Proof of Proposition 9.2. The first two terms in $\mathfrak{s}_{\pm}^{\#(i)}$ come from the formula (A.7). It is the sign of the conjugation of the determinant line associated with the restricted Pin boundary problem $\underline{D_{\mathbf{u}}'' \oplus D_0''}$ appearing in (8.12), induced from the conjugation on the moduli space of unmarked discs. The terms $k''+1+l''+\ell''$ account for conjugation on the configuration space of marked points, satisfying an extra incidence condition. The term $w_2(V)(\psi(d''))$ accounts for the change of orientation of the determinant line $\det(D_{\mathbf{u}}'' \oplus D_0'')$ arising from the change of Pin structure induced by the involution ϕ. Recall that the unique oriented path from $z \neq z_0$ to z' in the boundary of $\partial\hat{\Sigma}$, induced by the complex structure of $\hat{\Sigma}$, is very important for determining the canonical orientation of $\mathcal{L}^{\#'}$. This path will reverse under conjugation. The terms $\Upsilon^{(1)}$ (resp. $\Upsilon^{(2)} + w_1(d_b')w_1(\partial d'')$) in $\mathfrak{s}^{\#(1)}$ (resp. $\mathfrak{s}^{\#(2)}$) reflect this dependence. Note that reordering of the boundary marked points will affect the isomorphism in Definition 8.5 when the dimension of L is even. The last terms account for this dependence. $\square$

10 Equivalent definition of relatively open invariants

Recall from (8.2) that we have the parameterized moduli space of V-regular maps $\widetilde{\mathcal{M}}_{\mathbf{k},l,\mathbb{1}}^{V,\mathfrak{s}}(X, L, \mathbf{d})$, then we can use the method by Solomon to give a modified definition of the nonparameterized moduli space which is a section of the reparameterization group action. For instance, suppose $\Sigma = D^2$ and $l \geq 2$. Let

$$\pi_j : \widetilde{\mathcal{M}}_{\mathbf{k},l,\mathbb{1}}^{V,\mathfrak{s}}(X, L, \mathbf{d}) \to \Sigma$$

$$(u, \vec{z}, \vec{w}, \vec{q}) \mapsto w_j$$

be the projection to the j^{th} interior marked point. Take an interior point $\sigma_0 \in D^2$ and a line $\mathfrak{L} \subset D^2$ connecting σ_0 to ∂D^2 such that for any pair of points $(w, w') \in \Sigma \times \Sigma$ there exists a unique $\varphi \in \mathrm{Aut}(\Sigma) \simeq PSL_2(\mathbb{R})$ which satisfies

$$\varphi(w) = \sigma_0, \qquad \varphi(w') \in \mathfrak{L}.$$

Denote the distance function on D^2 by $d(\cdot,\cdot)$. Then we impose on the inhomogeneous term ν the condition

$$\nu(\cdot,\cdot,(w_1,w_2)) = \frac{d(w_1,w_2)}{d(w_1',w_2')}\nu(\cdot,\cdot,(w_1',w_2')). \tag{10.1}$$

Then we can define the nonparameterized moduli space as

$$\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\mathbf{d}) := (\pi_1\times\pi_2)^{-1}(\sigma_0\times\mathfrak{L}) \subset \widetilde{\mathcal{M}}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\mathbf{d}). \tag{10.2}$$

For general Σ, the nonparameterized moduli space can be similarly defined. Moreover, if there is no ϕ-multiply covered maps with non-positive Maslov index, the standard transversality arguments show that for a generic choice of (J,ν) under some constraints like (10.1), $\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\mathbf{d})$ is a smooth manifold of expected dimension. We will not show all constructions here and refer to [So] for an explanation of condition (10.1) and the similar constructions of nonparameterized moduli space of pseudoholomorphic (non-relatively) open maps in other cases.

From the discussion before, we see that we have two moduli spaces of V-regular maps $\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\mathbf{d})$ and, corresponding to the anti-symplectic involution ϕ, $\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\tilde{\mathbf{d}})$. And we can restrict the two total evaluation maps (8.3) and (9.1) to have

$$\mathbf{ev}:\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\mathbf{d}) \to L^{|\mathbf{k}|}\times X^l\times V^{\ell},$$

$$\widetilde{\mathbf{ev}}:\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\tilde{\mathbf{d}}) \to L^{|\mathbf{k}|}\times X^l\times V^{\ell}.$$

For generic choice of points $\vec{x}=(x_{ai})$, $x_{ai}\in L$, and pairs of points $\vec{y}_+=(y_j^+)$, $\vec{y}_-=(y_j^-)$ such that $y_j^+=\phi(y_j^-)$, $j=1,\cdots,l$, and $\vec{\mathbb{q}}_+=(\mathbb{q}_\jmath^+)$, $\vec{\mathbb{q}}_-=(\mathbb{q}_\jmath^-)$ such that $\mathbb{q}_\jmath^+=\phi(\mathbb{q}_\jmath^-)$, $\mathbb{q}_\jmath^\pm\in V$, $\jmath=1,\cdots,\mathbb{l}$, the two total evaluation maps will be transverse to

$$\prod_{a,i}x_{ai}\times\prod_j y_j^+\times\prod_\jmath \mathbb{q}_\jmath^+ \quad\text{and}\quad \prod_{a,i}x_{ai}\times\prod_j y_j^-\times\prod_\jmath \mathbb{q}_\jmath^-$$

in $L^{|\mathbf{k}|}\times X^l\times V^{\ell}$, respectively. For defining invariants, by index theorem, the following dimension condition would be satisfied

$$(n-1)(|\mathbf{k}|+2l)+(n-2)\cdot 2\ell = n(1-g)+\mu(d)-2\deg\mathbb{s}-\dim Aut(\Sigma), \tag{10.3}$$

where $\mu: H_2(X,L)\to\mathbb{Z}$ denote the Maslov index, g denote the genus of the closed Riemann surface $\Sigma\cup_{\partial\Sigma}\bar{\Sigma}$ obtained by doubling Σ. Note that $\mu(d)=\mu(\tilde{d})$, we can define a number as

$$M(V,\mathbf{d},\phi,\mathbf{k},l,\mathbb{l}) = \#\mathbf{ev}^{-1}(\vec{x},\vec{y}_+,\vec{\mathbb{q}}_+)+\#\widetilde{\mathbf{ev}}^{-1}(\vec{x},\vec{y}_-,\vec{\mathbb{q}}_-), \tag{10.4}$$

where $\#$ denotes the signed count with the sign of a given point, for example $\mathbf{u}\in\mathbf{ev}^{-1}(\vec{x},\vec{y}_+,\vec{\mathbb{q}}_+)$, depending on whether or not the isomorphism

$$d\mathbf{ev}_{\mathbf{u}}:\det(T\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\mathbf{d}))_{\mathbf{u}} \xrightarrow{\sim} \mathbf{ev}^*\det(T(L^{|\mathbf{k}|}\times X^l\times V^{\ell}))_{\mathbf{u}}$$

agrees with the underlying canonical isomorphism appearing in the Theorem 1.1

$$\det(T\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\mathbf{d}))\widetilde{\longrightarrow}\bigotimes_{a,i} evb^*_{ai}\det(TL),$$

up to the action of $\mathbb{R}^+$.

In particular, if $d=\tilde{d}$ we just simply define the number

$$\mathscr{R}\mathscr{N}:=\mathscr{R}\mathscr{N}(V,\mathbf{d},\mathbf{k},l,\mathbb{l})=M(V,\mathbf{d},\phi,\mathbf{k},l,\mathbb{l})=\#\mathbf{ev}^{-1}(\vec{x},\vec{y},\vec{\mathbb{q}}),\tag{10.5}$$

where $(\vec{x},\vec{y},\vec{\mathbb{q}})$ is a real configuration, *i.e.* $l=2c$, $\vec{y}=\{y^+_1,\cdots,y^+_c,y^-_1,\cdots,y^-_c\}$ satisfying $y^+_{j'}=\phi(y^-_{j'})$, $j'=1,\cdots,c$; and $\mathbb{l}=2\mathbb{c}$, $\vec{\mathbb{q}}=\{\mathbb{q}^+_1,\cdots,\mathbb{q}^+_{\mathbb{c}},\mathbb{q}^-_1,\cdots,\mathbb{q}^-_{\mathbb{c}}\}$ satisfying $\mathbb{q}^+_{\jmath'}=\phi(\mathbb{q}^-_{\jmath'})$, $\jmath'=1,\cdots,\mathbb{c}$.

Remark. For such special $d\in H_2(X,L)$, we can prove that $\mathscr{R}\mathscr{N}$ is an invariant if L is orientable and $\dim L\leq 3$ or if L might not be orientable and $\dim L=2$. However, for general homology class d, we might not expect that $M(V,\mathbf{d},\phi,\mathbf{k},l,\mathbb{l})$ is invariant. In the sequel, we will construct invariants for general homology class d, the proof of invariance of $\mathscr{R}\mathscr{N}(V,\mathbf{d},\mathbf{k},l,\mathbb{l})$ can be considered as a special case of the proof below of the invariance of more general invariants.

In order to define relatively open invariants for general homology class $d\in H_2(X,L)$, let us introduce more necessary notations. We denote by $\mathbb{d}=d_{\mathbb{C}}$ the doubling of d. For any homology class $\beta\in H_2(X,L)$, denote

$$\bar{\beta}=(\beta,\beta_1,\cdots,\beta_m)\in H_2(X,L)\oplus H_1(L)^{\oplus m},$$

and we denote the set of $(\mathbf{k},l+\mathbb{l})$-real configurations by $\mathcal{R}:=\mathcal{R}(\vec{x},\vec{y},\vec{\mathbb{q}})$ which is

$$\left\{\begin{array}{l}\vec{r}=(\vec{x},\vec{\xi},\vec{\lambda}\,)=(x_{11},\cdots,x_{mk_m},\xi_1,\cdots,\xi_l,\lambda_1,\cdots,\lambda_{\mathbb{l}})\\ \quad\mid \xi_j=y^+_j \text{ or } \xi_j=y^-_j,\ j=1,\cdots,l;\lambda_\jmath=\mathbb{q}^+_\jmath \text{ or } \lambda_\jmath=\mathbb{q}^-_\jmath,\ \jmath=1,\cdots,\mathbb{l}.\end{array}\right\}.\tag{10.6}$$

Moreover, denote by $\mathbf{ev}_{(\beta,\vec{r})}$ the total evaluation map

$$\mathbf{ev}_{(\beta,\vec{r})}:\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\bar{\beta})\to L^{|\mathbf{k}|}\times X^l\times V^{\ell},$$

$$(u,\vec{z},\vec{w},\vec{q})\mapsto(\vec{x}=u(\vec{z}),\vec{\xi}=u(\vec{w}),\vec{\lambda}=u(\vec{q})).$$

Now, we can rewrite the number of (1.4) as

$$\mathcal{I}:=\mathcal{I}^{V,\,\mathbb{s}}_{X,\phi,g,\mathbb{d},\mathbf{k},l}(\vec{x},\vec{y},\vec{\mathbb{q}})=\sum_{\forall\beta:\beta_{\mathbb{C}}=\mathbb{d};\forall\vec{r}\in\mathcal{R}}\#\mathbf{ev}^{-1}_{(\beta,\vec{r})}(\vec{x},\vec{\xi},\vec{\lambda}\,)\tag{10.7}$$

To show that the definition (10.7) is independent of the choices of $\vec{x}$ and pairs $(\vec{y}_+,\vec{y}_-)$, $(\vec{\mathbb{q}}_+,\vec{\mathbb{q}}_-)$ is equivalent to proving the expression (1.4) is independent of the choices of $\det(TL)$-valued n-forms α_{ai}, pairs of $2n$-forms (γ^+_j,γ^-_j) and pairs of $(2n-2)$-forms $(\eta^+_\jmath,\eta^-_\jmath)$, where $\gamma^\pm_j$ represent the Poincaré dual of $y^\pm_j$ for $j=1,\cdots,l$, and $\eta^\pm_\jmath$ represent the Poincaré dual of $\mathbb{q}^\pm_\jmath$ for $\jmath=1,\cdots,\mathbb{l}$.

11 Proof of invariance

In the following, for the special case $L \cap V = \emptyset$, we will show that the numbers $\mathcal{I}$ (in particular, $\mathscr{RN}$) are invariants, provided L is orientable and $\dim L \leq 3$, if L is nonorientable, we suppose that $\dim L = 2$ and the number of boundary marked points satisfy some additional conditions.

Suppose that we are given different points of real configuration $(\vec{x'}, \vec{y'}_{\pm}, \vec{\mathbb{q}'}_{\pm})$ satisfying the same generic conditions.

Let us denote

$$\begin{gathered}
\mathbf{x} : [0,1] \to L^{|\mathbf{k}|}, \quad \mathbf{x}(0) = \vec{x}, \quad \mathbf{x}(1) = \vec{x'}, \\
\mathbf{y}^{\pm} : [0,1] \to X^{l}, \quad \mathbf{y}^{+}(t) = \phi(\mathbf{y}^{-}(t)), \\
\mathbf{y}^{\pm}(0) = \vec{y}_{\pm}, \quad \mathbf{y}^{\pm}(1) = \vec{y'}_{\pm}, \\
\mathbf{q}^{\pm} : [0,1] \to V^{\ell}, \quad \mathbf{q}^{+}(t) = \phi(\mathbf{q}^{-}(t)), \\
\mathbf{q}^{\pm}(0) = \vec{\mathbb{q}}_{\pm}, \quad \mathbf{q}^{\pm}(1) = \vec{\mathbb{q}'}_{\pm}, \\
\Xi : [0,1] \to X^{l}, \quad \Xi(0) = \vec{\xi}, \quad \Xi(1) = \vec{\xi'}, \\
\Lambda : [0,1] \to V^{\ell}, \quad \Lambda(0) = \vec{\lambda}, \quad \Lambda(1) = \vec{\lambda'},
\end{gathered}$$

moreover, we require that $\xi_j = y_j^{+}$ (resp. y_j^{-}) if and only if $\xi'_j = y_j'^{+}$ (resp. $y_j'^{-}$), $\lambda_j = \mathbb{q}_j^{+}$ (resp. $\mathbb{q}_j^{-}$) if and only if $\lambda'_j = \mathbb{q}_j'^{+}$ (resp. $\mathbb{q}_j'^{-}$). Denote the set of all paths by

$$\mathbf{R} := \mathbf{R}(\mathbf{x}, \mathbf{y}^{\pm}, \mathbf{q}^{\pm}) = \{(\mathbf{x}, \Xi, \Lambda)\}$$

And denote

$$\mathcal{W}(\mathbf{x}, \Xi, \Lambda, \bar{\beta}) := \mathcal{M}_{\mathbf{k},l,\mathbb{1}}^{V,\,\$}(X, L, \bar{\beta})_{\mathbf{ev}_{(\beta,\vec{r})}} \times_{(\mathbf{x}\times\Xi\times\Lambda)\circ\triangle} ([0,1]), \tag{11.1}$$

$$\mathcal{W} = \mathcal{W}(\mathbf{x}, \mathbf{y}^{\pm}, \mathbf{q}^{\pm}, \mathbb{d}) := \bigcup_{\substack{\beta : \beta_{\mathbb{C}} = \mathbb{d}, \\ (\mathbf{x},\Xi,\Lambda) \in \mathbf{R}}} \mathcal{W}(\mathbf{x}, \Xi, \Lambda, \bar{\beta}). \tag{11.2}$$

Note that $\mathcal{W}$ gives a smooth oriented cobordism between

$$\bigcup_{\forall \beta:\beta_{\mathbb{C}}=\mathbb{d};\forall\ \vec{r}\in\mathcal{R}(\vec{x},\vec{y},\vec{\mathbb{q}})} \mathbf{ev}_{(\beta,\vec{r})}^{-1}(\vec{x}, \vec{\xi}, \vec{\lambda}\ )$$

and

$$\bigcup_{\forall \beta:\beta_{\mathbb{C}}=\mathbb{d};\forall\ \vec{r'}\in\mathcal{R}(\vec{x'},\vec{y'},\vec{\mathbb{q}'})} \mathbf{ev}_{(\beta,\vec{r'})}^{-1}(\vec{x'}, \vec{\xi'}, \vec{\lambda'}\ ).$$

Since in general $\mathcal{W}$ is noncompact, in order to prove the invariance of $\mathcal{I} := \mathcal{I}_{X,\phi,g,\mathbb{d},\mathbf{k},l}^{V,\,\$}$, we must research the stable boundary $\partial_G \mathcal{W}$ arising from the Gromov compactification of $\mathcal{W}$.

Note that for any tuple $\bar{\beta} = (\beta, \beta_1, \cdots, \beta_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m}$ the boundary of V-stable compactification is

$$\partial\overline{\mathcal{M}}_{\mathbf{k},l,\mathbb{1}}^{V,\,\$}(X, L, \bar{\beta}) := \mathcal{M}_{\mathbf{k},\sigma,l,\rho,\mathbb{1},\varrho}^{V,\,\$}(\hat{\Sigma}, X, L, \bar{\beta}', \beta'', 0),$$

where $\bar{\beta}' = (\beta', \beta_1, \cdots, \beta_b', \cdots, \beta_m)$, $\beta' = [u_1(\Sigma)]$, $\beta'' = [u_2(D^2)] \in H_2(X, L)$, such that $\beta = \beta' + \beta''$, $\partial\beta'' + \beta_b' = \beta_b$.

In fact, the moduli space $\mathcal{M}^{V,\,\$}_{\mathbf{k},\sigma,l,\rho,\mathbb{I},\varrho}(\hat{\Sigma}, X, L, \bar{\beta}', \beta'', 0)$ can also be redefined as a section of the reparameterization group action on the parameterized moduli space defined in (8.11). The constructions of these moduli spaces are also relative versions similar to those constructed in [So]. For instance, suppose that $\Sigma \simeq D^2$ and $l \geq 2$. Note that $\nu = 0$ on bubble components, thus there exists an action of $PSL_2(\mathbb{R})$ on the parameterized space (8.11) given by

$$(\mathbf{u}', (u'', \vec{z}'', \vec{w}'', \vec{q}'')) \to (\mathbf{u}', (u'' \circ \psi, (\psi^{-1})^{k''}\vec{z}'', (\psi^{-1})^{l''}\vec{w}'', (\psi^{-1})^{\mathbb{I}''}\vec{q}'')),$$

where $\psi \in PSL_2(\mathbb{R})$. Let

$$\pi_0 : \widetilde{\mathcal{M}}^{V,\,\$}_{\mathbf{k},\sigma,l,\rho,\mathbb{I},\varrho}(\hat{\Sigma}, X, L, \bar{\beta}', \beta'', 0) \to \Sigma$$

$$\mathbf{u} \mapsto z_0'$$

be the projection to the point where the bubble attaches. Then if w' does not bubble off, similar to (10.2) we can define the nonparameterized moduli space as

$$\begin{aligned} \mathcal{M}^{V,\,\$}_{\mathbf{k},\sigma,l,\rho,\mathbb{I},\varrho}(\hat{\Sigma}, X, L, \bar{\beta}', \beta'', 0) &:= (\pi_1 \times \pi_2)^{-1}(\sigma_0 \times \mathfrak{L})/PSL_2(\mathbb{R}) \\ &\subset \widetilde{\mathcal{M}}^{V,\,\$}_{\mathbf{k},\sigma,l,\rho,\mathbb{I},\varrho}(\hat{\Sigma}, X, L, \bar{\beta}', \beta'', 0)/PSL_2(\mathbb{R}), \end{aligned} \tag{11.3}$$

Otherwise, if w' bubbles off, we can define

$$\mathcal{M}^{V,\,\$}_{\mathbf{k},\sigma,l,\rho,\mathbb{I},\varrho}(\hat{\Sigma}, X, L, \bar{\beta}', \beta'', 0) := (\pi_1 \times \pi_0)^{-1}(\sigma_0 \times (\mathfrak{L} \cap \partial\Sigma))/PSL_2(\mathbb{R}). \tag{11.4}$$

For another special case that $\Sigma \simeq D^2$ and $l = 1$, the moduli space can be defined as follows. We first choose a ϕ-anti-invariant pseudo-cycle (A, f) representing the Poincaré dual of the symplectic form ω. Then taking $\mathcal{M}^{V,\,\$}_{\mathbf{k},\sigma,2,\rho,\mathbb{I},\varrho}(\hat{\Sigma}, X, L, \bar{\beta}', \beta'', 0)$ as defined above, we can perturb (A, f) slightly so that the evaluation map

$$evi_2 : \mathcal{M}^{V,\,\$}_{\mathbf{k},\sigma,2,\rho,\mathbb{I},\varrho}(\hat{\Sigma}, X, L, \bar{\beta}', \beta'', 0) \to X$$

is transversal to (A, f). Thus, we define

$$\mathcal{M}^{V,\,\$}_{\mathbf{k},\sigma,1,\rho,\mathbb{I},\varrho}(\hat{\Sigma}, X, L, \bar{\beta}', \beta'', 0) := \frac{1}{\omega(d)}\mathcal{M}^{V,\,\$}_{\mathbf{k},\sigma,2,\rho,\mathbb{I},\varrho}(\hat{\Sigma}, X, L, \bar{\beta}', \beta'', 0) \times_X A. \tag{11.5}$$

Note from the lemma 4.4 of [So] that one can choose the pseudo-cycle (A, f) which does not intersect L.

For simplicity, denote

$$\mathcal{M}(\Sigma) = \mathcal{M}^{V,\,\$}_{\mathbf{k}'+e_b,l',\mathbb{I}'}(\Sigma, X, L, \bar{\beta}', 0),$$

$$\mathcal{M}(D^2) = \mathcal{M}^{V,\,\$}_{k''+1,l'',\mathbb{I}''}(D^2, X, L, \beta'', 0).$$

Then

$$\begin{aligned}
\partial\overline{\mathcal{M}}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{l}}(X,L,\bar{\beta})) &= \mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},\sigma,l,\rho,\mathbb{l},\varrho}(\hat{\Sigma},X,L,\bar{\beta}',\beta'',0) \\
&= \mathcal{M}(\Sigma)_{evb_0'} \times_{evb_0''} \mathcal{M}(D^2). \qquad (11.6)
\end{aligned}$$

We can generically choose $(\mathbf{x},\mathbf{y}^{\pm},\mathbf{q}^{\pm})$ such that $(\mathbf{x},\Xi,\Lambda)=[\mathbf{x},(\Xi',\Xi''),(\Lambda',\Lambda'')]\in\mathbf{R}$ is transverse to the total evaluation map

$$\mathbf{ev}_{(\beta'+\beta'',\vec{r})}:\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},\sigma,l,\rho,\mathbb{l},\varrho}(\hat{\Sigma},X,L,\bar{\beta}',\beta'',0)\to L^{|\mathbf{k}|}\times X^l\times V^{\ell}.$$

We denote by

$$\begin{aligned}
&\partial_G\mathcal{W}^b_{\sigma,\rho,\varrho,\beta=\beta'+\beta''}(\Xi'',\Lambda'') \\
&:=\mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},\sigma,l,\rho,\mathbb{l},\varrho}(\hat{\Sigma},X,L,\bar{\beta}',\beta'',0)_{\mathbf{ev}_{(\beta'+\beta'',\vec{r})}}\times_{[\mathbf{x}\times(\Xi',\Xi'')\times(\Lambda',\Lambda'')]\circ\triangle}[0,1]
\end{aligned} \qquad (11.7)$$

the boundary stratum of the cobordism $\mathcal{W}$ arising from Gromov compactification. Denote

$$\begin{aligned}
&\partial_G\mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta',\beta''} \\
&:=\partial_G\mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta'+\beta''}(\Xi'',\Lambda'')\bigcup\partial_G\mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta'+\widetilde{\beta''}}(\phi(\Xi''),\phi(\Lambda'')),
\end{aligned}$$

and

$$\partial_G\mathcal{W}^b_{\sigma,\rho,\varrho}=\bigcup_{\beta',\beta''\,:\,(\beta'+\beta'')_{\mathbb{C}}=\mathbb{d}}\partial_G\mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta',\beta''}\ .$$

Thus the total Gromov compactified boundary of $\mathcal{W}$ is

$$\partial_G\mathcal{W}=\bigcup_{\substack{b\in[1,m],\sigma\subset[1,k_b],\\ \rho\subset[1,l],\varrho\subset[1,\mathbb{l}]}}\partial_G\mathcal{W}^b_{\sigma,\rho,\varrho}\ . \qquad (11.8)$$

Then we have an induced involution map on the boundary of cobordism

$$\phi_\partial:\partial_G\mathcal{W}\to\partial_G\mathcal{W},$$

$$\partial_G\mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta'+\beta''}(\Xi'',\Lambda'')\to\partial_G\mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta'+\widetilde{\beta''}}(\phi(\Xi''),\phi(\Lambda'')).$$

We claim that this map is fixed point free. Indeed, a fixed point of ϕ_∂ can only be in the strata satisfying $\beta''=\widetilde{\beta''}$. And we assume that there is no nonconstant ϕ-multiply covered pseudoholomorphic disc. Since a ϕ-somewhere injective disc can not be a fixed point of ϕ_∂, the fixed point of ϕ_∂ could only be the map that a zero energy disc bubbled off. The process of such bubbling-off corresponds to an interior marked point moving to the boundary. However, from the definition (11.5) of moduli space we know that the marked point constrained away from L can not move to the boundary. So the contradiction implies that ϕ_∂ has no fixed point.

We below will show that ϕ_∂ is orientation reversing if

(i) L is orientable, $\dim L=3$; or

(ii) L might not be orientable, $\dim L = 2$ and if L is nonorientable we require the condition (8.5) is satisfied.

• Case (i). L is orientable and $\dim L = 3$

Since $\dim L = 3$, by Wu relations, L is Pin^-, so $w_2(\mathbb{V}) = 0$. Since L is orientable, $\mu(d'')$ is even. Note the assumption $L \cap V = \emptyset$. Therefore, by the formula (9.8), the sign of $\Psi^{\#'}$ can be simplified as

$$\mathfrak{s}^{\#} = \frac{\mu(\beta'')}{2} + \deg \mathbb{s}'' + k'' + 1 + l'' + \ell'' \qquad \text{mod } 2, \tag{11.9}$$

Moreover, from the definition of the moduli space above we can see that the map

$$\phi^{\#} : \mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},\sigma,l,\rho,\mathbb{l},\varrho}(\hat{\Sigma}, X, L, \bar{\beta}', \beta'', 0) \to \mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},\sigma,l,\rho,\mathbb{l},\varrho}(\hat{\Sigma}, X, L, \bar{\beta}', \widetilde{\beta''}, 0)$$

has the same sign.

Note that the involution ϕ acts on X reversing the orientation since $\phi^*\omega^3 = -\omega^3$. So ϕ_∂ acts non-trivially on l'' of the factors of X^l. Therefore, the sign of the map between the two fiber products

$$\phi_\partial : \partial_G \mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta'+\beta''}(\Xi'', \Lambda'') \to \partial_G \mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta'+\widetilde{\beta''}}(\phi(\Xi''), \phi(\Lambda''))$$

should be independent of l''.

On the other hand, let us consider general dimension $n = \dim L$ for the moment. Recall (5.3) the virtual dimension of V-regular moduli space is

$$\begin{aligned} \dim \mathcal{M}^{V,\,\mathbb{s}}_{\mathbf{k},l,\mathbb{k},\mathbb{l}}(X, L, \bar{\beta}) &= \mu(\beta) + n(1-g) + |\mathbf{k}| \\ &\quad + 2(l + \ell - \deg \mathbb{s}) - \dim Aut(\Sigma). \end{aligned}$$

The definition of invariants requires the following dimension condition

$$\mu(\beta) + n(1-g) + |\mathbf{k}| + 2(l + \ell - \deg \mathbb{s}) - \dim Aut(\Sigma) = (|\mathbf{k}| + 2l)n + 2\ell(n-1).$$

So we have

$$\mu(\beta) = (|\mathbf{k}| + 2l)(n-1) + 2\ell(n-2) + 2\deg \mathbb{s} - n(1-g) + \dim Aut(\Sigma). \tag{11.10}$$

We observe that if we restrict the evaluation map to $L^{k''} \times X^{l''} \times V^{\ell''}$, the image should be at least codimension one to have a nontrivial intersection. That is

$$\begin{aligned} \dim \mathcal{M}(D^2) &= n + \mu(\beta'') + (k'' + 2l'') + 2(\ell'' - \deg \mathbb{s}'') - 3 \\ &\geq (k'' + 2l'')n + 2\ell''(n-1) - 1, \end{aligned} \tag{11.11}$$

therefore

$$\mu(\beta'') \geq (k'' + 2l'')(n-1) + 2\ell''(n-2) + 2\deg \mathbb{s}'' + 2 - n. \tag{11.12}$$

Similarly, we have

$$\begin{aligned} \dim \mathcal{M}(\Sigma) &= n(1-g) + \mu(\beta') + (k' + 2l') + 2(\ell' - \deg \mathbb{s}') - \dim Aut(\Sigma) \\ &\geq (k' + 2l')n + 2\ell'(n-1) - 1, \end{aligned} \tag{11.13}$$

and so

$$\begin{aligned}\mu(\beta') &\geq (k'+2l')(n-1)+2\ell'(n-2)+2\deg \mathbb{s}' \\ &\quad + \dim Aut(\Sigma)-1-n(1-g). \end{aligned} \tag{11.14}$$

Then from (11.10) and (11.14) we have

$$\begin{aligned}\mu(\beta'') &= \mu(\beta)-\mu(\beta') \\ &\leq (k''+2l'')(n-1)+2\ell''(n-2)+2\deg \mathbb{s}''+1. \end{aligned} \tag{11.15}$$

In particular, when $n=3$, from (11.12) and (11.15), and noting that $\mu(\beta'')$ is even, we see that each set $\partial_G \mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta'+\beta''}(\Xi'',\Lambda'')$ is nonempty if and only if the following dimension condition is satisfied

$$\mu(\beta'')=2(k''+2l'')+2\ell''+2\deg \mathbb{s}''. \tag{11.16}$$

Thus we have

$$sign(\phi_\partial)=\frac{\mu(\beta'')}{2}+\deg \mathbb{s}''+k''+1+\ell''\cong 1 \pmod 2.$$

That is to say, the map ϕ_∂ reverses orientation. That means $\#\partial_G\mathcal{W}=0$. Therefore, we have

$$\begin{aligned} 0=\#\partial\overline{\mathcal{W}} &= \sum_{\beta;\vec{r'}}\#\mathbf{ev}^{-1}_{(\beta,\vec{r'})}(\vec{x'},\vec{\xi'},\vec{\lambda'}\ )-\sum_{\beta;\vec{r}}\#\mathbf{ev}^{-1}_{(\beta,\vec{r})}(\vec{x},\vec{\xi},\vec{\lambda}\ )+\#\partial_G\mathcal{W} \\ &= \mathcal{I}^{V,\ \mathbb{s}}_{X,\phi,g,\mathbb{d},\mathbf{k},l}(\vec{x'},\vec{y'},\vec{\mathbb{q}'})-\mathcal{I}^{V,\ \mathbb{s}}_{X,\phi,g,\mathbb{d},\mathbf{k},l}(\vec{x},\vec{y},\vec{\mathbb{q}}). \end{aligned}$$

So integers $\mathcal{I}^{V,\ \mathbb{s}}_{X,\phi,g,\mathbb{d},\mathbf{k},l}$ are independent of the choice of $(\vec{x},\vec{y},\vec{\mathbb{q}})$. Equivalently, we can say that the integral in (1.4) is independent of the choices of α_{ai}, γ_j and $\eta_{\jmath}$. Similarly, we can prove that $\mathcal{I}^{V,\ \mathbb{s}}_{X,\phi,g,\mathbb{d},\mathbf{k},l}$ are independent of the generic choice of $J\in\mathcal{J}_{\omega,\phi}$, and the choice of inhomogeneous perturbation $\nu\in\mathcal{P}_{\phi,c}$. That means $\mathcal{I}^{V,\ \mathbb{s}}_{X,\phi,g,\mathbb{d},\mathbf{k},l}$ are invariants of the tuple (X,ω,V,ϕ). In particular, if $d=\tilde{d}$, the numbers $\mathscr{R}\mathscr{N}=M(V,\mathbf{d},\phi,\mathbf{k},l,\mathbb{1})$ in (10.5) are also invariants for this case.

• Case (ii). $\dim L=2$, and (8.5) is satisfied if L is not orientable

Also by Wu relations, L is Pin^-, so $w_2(\mathbb{V})=0$. As the argument above, we conclude that the sign of the map $\phi^{\#}$ is given by formulas (9.6) and (9.7) depending on whether or not $1\in\sigma$. Since $\phi^*\omega^2=\omega^2$, ϕ preserves the orientation of X. It means that (9.6) and (9.7) also coincide with the signs of the map ϕ_∂. Then by inequalities (11.12) and (11.15), note $n=2$, we see that the stratum $\partial_G\mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta',\beta''}$ will be empty unless

$$\mu(\beta'')+r=k''+2l''+2\deg \mathbb{s}'' \tag{11.17}$$

for $r=0$ or -1. The following calculation shows that the signs (9.6) and (9.7) are exactly 1.

First, assume $1 \notin \sigma$. Using the restriction (11.17), we have

$$\begin{aligned}
\frac{k''(k''-1)}{2} &= \frac{(\mu(\beta'')+r-2l''-2\deg \mathbf{s}'')(\mu(\beta'')+r-2l''-2\deg \mathbf{s}''-1)}{2} \\
&\cong \frac{\mu(\beta'')(\mu(\beta'')-1)}{2}+\frac{r(r-1)}{2}+l''+\deg \mathbf{s}''+r\mu(\beta'') \pmod 2. \quad (11.18)
\end{aligned}$$

Also from (11.17), we have

$$\begin{aligned}
\Upsilon^{(1)}(\beta'',k'') \cong \mu(\beta'')k'' &\cong \mu(\beta'')^2+r\mu(\beta'')+2(l''+\deg \mathbf{s}'')\mu(\beta'') \\
&\cong \mu(\beta'')^2+r\mu(\beta'') \pmod 2. \quad (11.19)
\end{aligned}$$

Substituting (11.17), (11.18) and (11.19) into (9.6), we have

$$\mathfrak{s}_-^{\#(1)} \cong \frac{r(r+1)}{2}+\ell''+1 \pmod 2.$$

Also note that when $n=2$, the involution ϕ acts on 1 dimensional submanifold V reversing the orientation since $\phi^*\omega=-\omega$. So ϕ_∂ acts non-trivially on ℓ'' of the factors of V^ℓ. Therefore, the sign of the map between the two fiber products

$$\phi_\partial : \partial_G \mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta'+\beta''}(\Xi'',\Lambda'') \to \partial_G \mathcal{W}^b_{\sigma,\rho,\varrho,\ \beta'+\widetilde{\beta''}}(\phi(\Xi''),\phi(\Lambda''))$$

should be independent of ℓ''. Thus we have

$$sign(\phi_\partial) = \frac{r(r+1)}{2}+1 \cong 1 \pmod 2$$

since $r=0$ or -1.

Then we consider the case $1 \in \sigma$. Using the restriction (11.17) again, we have

$$\begin{aligned}
\frac{(k''-1)(k''-2)}{2} &= \frac{(\mu(\beta'')+r-2l''-2\deg \mathbf{s}''-1)(\mu(\beta'')+r-2l''-2\deg \mathbf{s}''-2)}{2} \\
&\cong \frac{\mu(\beta'')(\mu(\beta'')+1)}{2}+\frac{r(r+1)}{2} \\
&\quad +l''+\deg \mathbf{s}''+r\mu(\beta'')+1 \pmod 2. \quad (11.20)
\end{aligned}$$

Recall the condition (8.5): $w_1(\beta_b) \cong k_b+1$ and (11.17), we calculate

$$k_b(k''+1) \cong (w_1(\beta_b)+1)(\mu(\beta'')+r+1) \pmod 2. \quad (11.21)$$

Substituting (11.17), (11.20) and (11.21) into (9.7), noting that $r=0$ or -1, we have

$$\begin{aligned}
\mathfrak{s}_-^{\#(2)} &\cong \frac{r(r+1)}{2}+r\mu(\beta'')+r+\ell'' \\
&\quad +(w_1(\beta_b)+1)(\mu(\beta'')+r+1)+\Upsilon^{(2)}+w_1(\beta_b')w_1(\partial\beta'') \\
&\cong r\mu(\beta'')+r+\ell''+(w_1(\beta_b)+1)(\mu(\beta'')+r+1) \\
&\quad +\Upsilon^{(2)}+w_1(\beta_b')w_1(\partial\beta'') \pmod 2. \quad (11.22)
\end{aligned}$$

Recall the formula (9.5) and the fact

$$w_1(\beta_b) = w_1(\beta_b') + w_1(\partial\beta''),$$

we can express

$$\Upsilon^{(2)} = w_1(\beta_b)(k'' - 1) + w_1(\partial\beta'')k'. \tag{11.23}$$

Considering the fact that $\mu(\beta'') \cong w_1(\partial\beta'')$ (mod 2), and using (11.17) and (8.5), we obtain

$$\begin{aligned} w_1(\partial\beta'')k' &\cong w_1(\partial\beta'')(k_b - k'') \\ &\cong w_1(\partial\beta'')(w_1(\beta_b) + 1 + \mu(\beta'' + r)) \\ &\cong w_1(\partial\beta'')w_1(\beta_b') + \mu(\beta'')(1 + r) \qquad (\text{mod } 2). \end{aligned} \tag{11.24}$$

Substituting (11.24) and (11.17) in (11.23), we get

$$\Upsilon^{(2)} \cong w_1(\beta_b)(\mu(\beta'') + r + 1) + w_1(\partial\beta'')w_1(\beta_b') + \mu(\beta'')(1 + r) \pmod 2.$$

Substituting the last formula in (11.22), we calculate

$$\begin{aligned} \mathfrak{s}_-^{\#(2)} &\cong r\mu(\beta'') + r + \ell'' + (\mu(\beta'') + r + 1) + \mu(\beta'')(1 + r) \\ &\cong \ell'' + 1 \qquad\qquad (\text{mod } 2). \end{aligned} \tag{11.25}$$

As the mentioned reason above, the sign of ϕ_∂ is independent of ℓ'', so $sign(\phi_\partial) \cong 1 \pmod 2$, which implies $\mathcal{I}^{V,\,\mathfrak{s}}_{X,\phi,g,\mathbb{d},\mathbf{k},l}$ are invariants of the tuple (X, ω, V, ϕ). In particular, the numbers $\mathscr{R}\mathscr{N} = M(V, \mathbf{d}, \phi, \mathbf{k}, l, \mathbb{1})$ in (10.5) are invariants of the tuple (X, ω, V, ϕ).

Appendix

For the convenience of the reader, we review some definitions and important conclusions in [So] about the orientation of determinant of real linear Cauchy-Riemann operator. For our concrete problem of intersection of stable maps with a codimensional two symplectic submanifold, we state that similar conclusions hold for some kind of restriction of Cauchy-Riemann operator.

We denote by Γ an appropriate Banach space completion of the smooth section of a vector bundle. For a vector bundle $V \to B$, we denote by $\mathfrak{F}(V)$ the orthonormal frame bundle of V which is a principal $O(n)$ bundle. Recall the fact that the Lie group $Spin(n)$ is the central $\mathbb{Z}/2\mathbb{Z}$ extension of the special orthogonal group $SO(n)$, similarly, the two groups $Pin^+(n)$ and $Pin^-(n)$, although are topologically the same, are two different central extensions of $O(n)$.

Definition A.1 *A $Pin^\pm(n)$ structure $\mathfrak{P} = (P, p)$ on a vector bundle $V \to B$ consists of principal $Pin^\pm(n)$ bundle $P \to B$ and a $Pin^\pm(n) - O(n)$ equivariant bundle map*

$$p : P \to \mathfrak{F}(V).$$

A map $\varphi : V \to V'$ between vector bundles with Pin structure preserves Pin structure if there exists a lifting $\tilde{\varphi}$ such that the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "\tilde{\varphi}"] \arrow[d, "p"] & P' \arrow[d, "p'"] \\ \mathfrak{F}(V) \arrow[r, "\varphi"] & \mathfrak{F}(V') \end{tikzcd}$$

The obstruction to putting a *Spin* structure on a bundle $V \to B$ is $w_2(V) \in H^2(B, \mathbb{Z}/2\mathbb{Z})$. For Pin^+ it is still $w_2(V)$, and for Pin^- it is $w_2(V) + w_1^2(V)$.

Definition A.2 *Let $(\Sigma, \partial\Sigma)$ be a Riemann surface with boundary $\partial\Sigma = \coprod_{a=1}^m (\partial\Sigma)_a$. A Cauchy-Riemann (or Riemann-Roch) Pin boundary value problem*

$$\underline{D} = (\Sigma, E, F, \mathfrak{P}, D)$$

consists of

1° *a complex vector bundle $E \to \Sigma$,*

2° *a totally real sub-bundle over the boundary $F \to \partial\Sigma$,*

3° *a Pin^+ or Pin^- structure $\mathfrak{P}$ on F,*

4° *an orientation of $F|_{(\partial\Sigma)_a}$ for each a so that $F|_{(\partial\Sigma)_a}$ is orientable,*

5° *a differential operator*

$$D : \Gamma((\Sigma, \partial\Sigma), (E, F)) \to \Gamma(\Sigma, \Omega^{0,1}(E))$$

such that for $f \in C^\infty(\Sigma, R)$, $\xi \in \Gamma((\Sigma, \partial\Sigma), (E, F))$,

$$D(f\xi) = fD\xi + (\bar{\partial} f)\xi.$$

D is called a real linear Cauchy-Riemann operator.

Remark. For the sake of studying the (J, ν)-holomorphic maps relative to a codimensional 2 symplectic submanifold $V \subset X$, say V-regular maps, we may consider the restriction of a real linear Cauchy-Riemann operator D to a subspace $\Gamma^{rest} := \Gamma_{(\mathbb{r},\mathbb{s})}((\Sigma, \partial\Sigma), (E, F))$. A section $\xi \in \Gamma^{rest}$ (or $\Gamma_{(\mathbb{r},\mathbb{s})}$) if and only if it satidfies some vanishing conditions at each prescribed (say, intersection) marked point $p_{a\imath}$ (resp. $q_\jmath$). In the present paper, we only consider the orientability of moduli space under the condition $L \cap V = \emptyset$, thus we denote the subspace simply by Γ^{rest} or $\Gamma_{\mathbb{s}}$ and the restriction of D to this subspace by D^{rest} or $D_{\mathbb{s}}$. In particular, for the concrete case of moduli space, $D_{\mathbb{s}}$ is the restriction by contact condition of $D_u = D_u\bar{\partial}_{(J,\nu)}$, where $u \in \mathcal{M}^V(X, L, d)$. When the pair $(J, \nu) \in \mathbb{J}^V$ (or $\mathbb{J}^V_\phi$) is V-compatible (see Definition 4.1), the arguments in Lemma 5.1 show that the restriction is transverse. Therefore, in such concrete situation $D_{\mathbb{s}}$ is D_u for $u \in \mathcal{M}^V_{\mathbb{s}}(X, L, d)$.

It is not difficult to apply the arguments by Mcduff-Salamon (see [McS] Appendix C.2) to show that D^{rest} is a Fredholm operator. We call $\underline{D^{rest}} = (\Sigma, E, F, \mathfrak{P}, D^{rest})$ the *restricted Pin boundary value problem.*

For a Fredholm operator D, we define its determinant line by

$$\det(D) := \Lambda^{\max}(\ker D) \otimes \Lambda^{\max}(\text{coker } D)^*.$$

As explained in [McS], for a family of Fredholm operators, $\det(D)$ denotes a line bundle with the natural topology.

Since we may trivialize E over Σ and each component of boundary $(\partial\Sigma)_a \simeq S^1$, the restriction of F to each $(\partial\Sigma)_a$ defines a loop of totally real subspaces of $\mathbb{C}^n$. it is well known such a loop associates a Maslov index μ_a. we denote by

$$\mu(E,F) = \sum_{a=1}^{m} \mu_a \tag{A.1}$$

the total Maslov index of the vector bundle pair (E,F). Moreover, $\mu(E,F)$ doesn't depend on the choice of trivialization of E.

We say $\underline{\varphi} : \underline{D} \to \underline{D}'$ is an *isomorphism* of Cauchy-Riemann (or restricted) *Pin* boundary value problems if

i) there exists a biholomorphic map $f : \Sigma \to \Sigma'$;

ii) there exists an isomorphism of bundles $\varphi : E \to E'$ covering f such that $\varphi|_{\partial\Sigma}$ maps F to F' and preserving *Pin* structure and preserving orientation if F, F' are orientable;

iii) $\varphi \circ D = D' \circ \varphi$.

J. Solomon showed the following (Proposition 2.8 and Lemma 2.9 in [So])

Proposition A.1 *The determinant line of a real-linear Cauchy-Riemann Pin boundary value problem* $\underline{D}$ *admits a canonical orientation. If* $\underline{\varphi} : \underline{D} \to \underline{D}'$ *is an isomorphism, then the induced morphism*

$$\Psi : \det(D) \to \det(D')$$

preserves the canonical orientation. Furthermore, the canonical orientation varies continuously in a family of Cauchy-Riemann operators. That is, it defines a single component of the determinant line bundle over that family. If the boundary condition $F|_{(\partial\Sigma)_a}$ *is orientable, then reversing the orientation on* $F|_{(\partial\Sigma)_a}$ *will change the canonical orientation on* $\det(D)$.

Remark. It is not difficult to apply the method of proof by Solomon to generalize the Proposition above to the case for the restricted *Pin* boundary value problem $\underline{D^{rest}}$. For instance, it is important in the proof of Proposition 2.8 in [So] that, for any chosen Cauchy-Riemann operator $\tilde{D}$ on the restriction $\hat{E}|_{\tilde{\Sigma}}$, $\det(\tilde{D})$ has the canonical complex orientation, where $\hat{E} \to \hat{\Sigma}$ is the degenerated vector bundle of $E \to \Sigma$, $\hat{\Sigma} = \tilde{\Sigma} \cup_a \Delta_a$ ($\tilde{\Sigma}$ is the closed component, Δ_a is a disk corresponding to each $(\partial\Sigma)_a$). In our relative case, we consider the restriction $\tilde{D}^{rest}$ which is a complex Fredholm operator, and $\det(\tilde{D}^{rest})$ also can be equipped with the canonical complex orientation. The remain arguments in the proof of Proposition 2.8 in [So] can go through with minor modifications.

We then come to study the sign of conjugation on the canonical orientation of the determinant line of a Cauchy-Riemann (or restricted) *Pin* boundary value problem. More precisely, given a Riemann surface Σ, let $\bar{\Sigma}$ denote the same topological surface with conjugate complex structure, and let

$$c : \Sigma \to \bar{\Sigma}$$

denote the tautological anti-holomorphic map. Similarly, let $(\bar{E}, \bar{F})$ denote the same real bundle pair with the opposite complex structure on E, and denote by

$$C : E \to \bar{E}$$

the tautological anti-complex-linear bundle map. Furthermore, a Cauchy-Riemann (resp. restricted) operator D (resp. $D_{\mathbb{s}}$) on the bundle $E \to \Sigma$ is the same as a Cauchy-Riemann (resp. restricted) operator $\bar{D}$ (resp. $\bar{D}_{\mathbb{s}}$) on the bundle $\bar{E} \to \bar{\Sigma}$. So, given any Cauchy-Riemann (resp. restricted) *Pin* boundary problem $\underline{D}$ (resp. $\underline{D_{\mathbb{s}}}$), we may construct its conjugate $\underline{\bar{D}}$ (resp. $\underline{\bar{D}_{\mathbb{s}}}$). Clearly, we have a tautological map of Cauchy-Riemann (resp. restricted) *Pin* boundary value problems

$$\underline{C} : \underline{D} \to \underline{\bar{D}} \ \ \text{(resp. } \underline{D_{\mathbb{s}}} \to \underline{\bar{D}_{\mathbb{s}}}\text{)}.$$

Suppose the genus of $\Sigma/\partial\Sigma$ is g_0, the number of boundary components of Σ is m and $\mathrm{rank}(F) = n$. Denote by $\mu = \mu(E, F)$ and by w_1 the first Stiefel-Whitney class. Solomon calculated the sign of the induced isomorphism $\Psi : \det(D) \to \det(\bar{D})$ (Proposition 2.12 in [So]).

Proposition A.2 *The sign of the induced isomorphism Ψ relative to the respective canonical orientation is given by*

$$\begin{aligned} sign^{+}(\underline{D}) \ &= \ \frac{\mu(\mu+1)}{2} + (1-g_0)n + mn \\ &\quad + \sum_{a<b} w_1(F)((\partial\Sigma)_a) w_1(F)((\partial\Sigma)_b) \quad \mathrm{mod}\ 2, \end{aligned} \tag{A.2}$$

for a Pin^{+} structure and

$$\begin{aligned} sign^{-}(\underline{D}) \ &= \ \frac{\mu(\mu+1)}{2} + (1-g_0)n + mn \\ &\quad + \sum_{a<b} w_1(F)((\partial\Sigma)_a) w_1(F)((\partial\Sigma)_b) \\ &\quad + \sum_{a=1}^{m} w_1(F)((\partial\Sigma)_a) \qquad\qquad \mathrm{mod}\ 2, \end{aligned} \tag{A.3}$$

for a Pin^{-} structure. In particular, when $\Sigma = D^2$, $g_0 = 0$ and $m = 1$ we have

$$sign^{\pm}(\underline{D}) = \frac{\mu(\mu \pm 1)}{2} \qquad \mathrm{mod}\ 2. \tag{A.4}$$

As to our concrete case of regular maps, for the restricted *Pin* boundary value problem $\underline{D_{\mathbb{s}}}$, we study the sign of the induced isomorphism $\Psi : \det(D_{\mathbb{s}}) \to \det(\bar{D}_{\mathbb{s}})$. Recall that when the pair $(J, \nu) \in \mathbb{J}^V$ (or $\mathbb{J}^V_{\phi}$), we can consider $D_{\mathbb{s}} = D_u$ for $u \in \mathcal{M}^V_{\mathbb{s}}(X, L, d)$, where $\mathbb{s} = (s_1, \cdots, s_{\mathbb{l}})$ is the list of multiplicities of interior intersection points of a V-regular map, and $\deg \mathbb{s} = \sum_{\jmath=1}^{\mathbb{l}} s_{\jmath}$. The argument is a modification of the one of Proposition 2.12 in [So], involving the interior contact conditions. Basically, one need modify the formula

(9) in [So] respecting to our concrete case of V-regular (J,ν)-maps for generic $(J,\nu)\in\mathbb{J}^V$, the transversality of contact conditions (see Lemma 5.1) implies the relation

$$\mathrm{index}_{\mathbb{C}}(\tilde{D}_{\mathbb{s}})=\mathrm{index}_{\mathbb{C}}(\tilde{D})-\deg\,\mathbb{s},$$

where $\tilde{D}_{\mathbb{s}}$ is the restriction of the chosen complex Cauchy-Riemann operator $\tilde{D}$ on the restriction $\hat{E}|_{\tilde{\Sigma}}$. Thus, we have

Proposition A.3 *The sign of the induced isomorphism $\Psi:\ \det(D_{\mathbb{s}})\to\det(\bar{D}_{\mathbb{s}})$ relative to the respective canonical orientation is given by*

$$\begin{aligned} sign^{+}(\underline{D_{\mathbb{s}}}) &= \frac{\mu(\mu+1)}{2}+\deg\,\mathbb{s}+(1-g_0)n+mn \\ &\quad+\sum_{a<b}w_1(F)((\partial\Sigma)_a)w_1(F)((\partial\Sigma)_b) \mod 2, \end{aligned} \tag{A.5}$$

for a Pin^{+} structure and

$$\begin{aligned} sign^{-}(\underline{D_{\mathbb{s}}}) &= \frac{\mu(\mu+1)}{2}+\deg\,\mathbb{s}+(1-g_0)n+mn \\ &\quad+\sum_{a<b}w_1(F)((\partial\Sigma)_a)w_1(F)((\partial\Sigma)_b) \\ &\quad+\sum_{a=1}^{m}w_1(F)((\partial\Sigma)_a) \mod 2, \end{aligned} \tag{A.6}$$

for a Pin^{-} structure. In particular, when $\Sigma=D^2$, $g_0=0$ and $m=1$ we have

$$sign^{\pm}(\underline{D_{\mathbb{s}}})=\frac{\mu(\mu\pm1)}{2}+\deg\,\mathbb{s} \qquad \mod 2. \tag{A.7}$$

Definition A.3 *A short exact sequence of families of Fredholm operators*

$$0\to D'\to D\to D''\to 0$$

consists of a base parameter space B, two short exact sequences of Banach space bundles over B

$$0\to X'\to X\to X''\to 0,\qquad 0\to Y'\to Y\to Y''\to 0.$$

and three Fredholm morphisms of Banach bundles

$$D:X\to Y,\qquad D':X'\to Y'\qquad D'':X''\to Y'',$$

such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0 \\ & & D'\uparrow & & D\uparrow & & D''\uparrow & & \\ 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \end{array}$$

The following lemma will be used in section 8.

Lemma A.1 *A short exact sequence of families of Fredholm operators*

$$0\to D'\to D\to D''\to 0$$

induces an isomorphism

$$\det(D')\otimes\det(D'')\simeq\det(D).$$

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